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Twistor Transform in d Dimensions and a Unifying Role for Twistors

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Abstract

Twistors in four dimensions $d = 4$ have provided a convenient description of massless particles with any spin, and this led to remarkable computational techniques in Yang-Mills field theory. Recently it was shown that the same $d = 4$ twistor provides also a unified description of an assortment of other particle dynamical systems, including special examples of massless or massive particles, relativistic or non-relativistic, interacting or non-interacting, in flat space or curved spaces. In this paper, using 2T-physics as the primary theory, we derive the general twistor transform in d-dimensions that applies to all cases, and show that these more general twistor transforms provide d dimensional holographic images of an underlying phase space in flat spacetime in $d + 2$ dimensions. Certain parameters, such as mass, parameters of spacetime metric, and some coupling constants appear as moduli in the holographic image while projecting from $d + 2$ dimensions to $(d - 1) + 1$ dimensions or to twistors. We also extend the concept of twistors to include the phase space of D-branes, and give the corresponding twistor transform. The unifying role for the same twistor that describes an assortment of dynamical systems persists in general, including D-branes. Except for a few special cases in low dimensions that exist in the literature, our twistors are new.

I. BOTTOM→UP APPROACH TO SPACETIME

In this section we will discuss mainly spinless particles to establish some concepts in a familiar setting. A similar analysis applies to particles with spin[1], therefore when we discuss twistors in four dimensions, $d = 4$ in section-I B, we include the spin in some of the discussion. Generalizations of the twistors to higher dimensions, and D-branes, will be presented in the following sections. Other generalizations, including spinning particles, supersymmetry, compactified internal spaces, will be given elsewhere.

A. Phase space

A massless and spinless relativistic particle in $(d-1)$ space dimensions is described by its position-momentum phase space coordinates $(\vec{x}^i(t), \vec{p}^i(t))$, $i = 1, 2, \cdots, (d-1)$ while its time development is governed by the Hamiltonian $H = |\vec{p}| = \sqrt{\vec{p}^2}$. Hence its action is

$$S = \int dt \left(\partial_t \vec{x} \cdot \vec{p} - |\vec{p}|\right). \tag{1.1}$$

For a spinning particle we include spin degrees of freedom in an enlarged phase space. Of course, a massless particle is a relativistically invariant system, and this is verified by the fact that this action is invariant under Lorentz transformations $\mathrm{SO}(d-1,1)$. However, this symmetry is only partially manifest in this action: rotation symmetry $\mathrm{SO}(d-1)$ is evident while the boost symmetry is hidden. To make the Lorentz symmetry fully manifest one must introduce a gauge symmetry together with extra matter gauge degrees of freedom and a gauge field. This process is a first step in the bottom→up approach that helps us discover a deeper point of view of symmetries and their connection to spacetime.

The well known bottom-up approach in this example is to introduce the worldline reparametrization gauge symmetry and then use the larger phase space $(x^\mu(\tau), p^\mu(\tau))$ with $\mu = 0, 1, 2, \cdots, (d-1)$. The action in the first order formalism is

$$S(x,p) = \int d\tau \left(\partial_\tau x^\mu p_\mu - \frac{1}{2} e p_\mu p_\nu \eta^{\mu\nu}\right), \tag{1.2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $e(\tau)$ is the gauge field coupled to the generator of gauge transformations $p^2/2$. The gauge transformations

$$\delta_\varepsilon e = \partial_\tau \varepsilon(\tau), \quad \delta_\varepsilon x^\mu = \varepsilon(\tau) p^\mu, \quad \delta_\varepsilon p_\mu = 0, \tag{1.3}$$

[1] A detailed discussion of the spinning particle and the related generalized twistors, including higher dimensions and supersymmetry, will be given in a future paper [1].

transform the Lagrangian into an ignorable total derivative $\delta_\varepsilon S = \int d\tau \partial_\tau (\varepsilon p^2/2) = 0$. The action (1.2) has an evident global Lorentz symmetry due to the fact that all terms are Lorentz dot products. Noether's theorem gives the conserved charges $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$. These are gauge invariant $\delta_\varepsilon L^{\mu\nu} = 0$, and hence they are physical observables. So the $L^{\mu\nu}$ remain conserved, and act as the generators of symmetry of the gauge invariant action $S(x,p)$, even if some arbitrary gauge is fixed.

The relation between Eqs.(1.1,1.2) is obtained in a fixed gauge. This is the reverse process, namely it is part of the top→bottom approach that will be discussed in the next section. The equation of motion with respect to the gauge field e requires that the gauge generator vanishes $p^2{=}0$, implying that the physical sector (massless particle on mass shell) must be gauge invariant. The gauge symmetry can be fixed by taking $x^0(\tau) = t(\tau) = \tau$, and the constraint can be solved for the canonical conjugate to $x^0 = \tau$, namely $p^0 = \pm|\vec{p}|$. The remaining phase space $(\vec{x},\vec{p})$ provides a parametrization of the gauge invariant sector. Taking the positive root $p^0 = +|\vec{p}|$, we derive the non-covariant action Eq.(1.1) for the massless particle as the gauge fixed form of the gauge invariant Eq.(1.2). Similarly, the gauge fixed form of the gauge invariant $L^{\mu\nu}$, given by $L^{ij} = x^i p^j - x^j p^i$, $L^{0i} = \tau p^i - x^i|\vec{p}|$, are the generators of the non-linearly realized hidden global SO$(d-1,1)$ symmetry of the gauge fixed Lagrangian in Eq.(1.1). These generate the Lorentz transformations of phase space $(\vec{x}(\tau),\vec{p}(\tau))$ at any τ through the Poisson brackets $\delta_\omega \vec{x}(\tau) = \frac{1}{2}\omega_{\mu\nu}\{L^{\mu\nu},\vec{x}\}(\tau)$, and $\delta_\omega \vec{p}(\tau) = \frac{1}{2}\omega_{\mu\nu}\{L^{\mu\nu},\vec{p}\}(\tau)$ where τ is treated like a parameter. It can be checked that under these transformations the non-covariant looking action in Eq.(1.1) is Lorentz invariant (dropping an ignorable total derivative $\delta S = \int d\tau\ \partial_\tau (f(\tau)) = 0$).

The covariant formulation in Eq.(1.2) provides a greater flexibility to analyze the system from a broader and more fundamental perspective. For example, one may choose other gauges besides the timelike gauge $x^0(\tau) = \tau$ that relates Eqs.(1.1) and (1.2). In particular the lightcone gauge $x^+(\tau) = \tau$, in which the constraint $p^2 = 0$ is solved for the canonical conjugate $p^- = p_\perp^2/2p^+$, has certain advantages in computation.

One may also analyze the system covariantly. For example in covariant quantization one may apply the constraints on the physical states to derive the Klein-Gordon equation, and from it the Klein-Gordon free field theory

$$p^2|\varphi\rangle = 0 \text{ gauge invariant } \Leftrightarrow \text{ physical states.} \tag{1.4}$$

$$\langle x|p^2|\varphi\rangle = 0 = -\partial^\mu\partial_\mu\varphi(x) \ \to S_{KG} = \int d^4x\ \partial^\mu\varphi^*\partial_\mu\varphi. \tag{1.5}$$

The covariant formulation in Eq.(1.2) is one of the stations in the bottom→up approach toward a deeper point of view of symmetries and spacetime.

We are not done yet with the hidden symmetries of the non-covariant action in Eq.(1.1). This system has the larger symmetry SO$(d,2)$, namely conformal symmetry which is a

general symmetry of massless systems. This symmetry persists as a hidden symmetry in the covariant action in Eq.(1.2) and in the Klein-Gordon action in Eq.(1.5). It has been known that the SO$(d,2)$ symmetry can be made manifest in two ways: one is twistors [2][3] (in $d=4$), and the other is two time physics (2T-physics) [4][5] in any d. Actually these are related to each other by gauge transformations in the 2T-physics formalism [6][7] in the twistor gauge as we will discuss in the rest of the paper in more detail. Either way, the route to making the SO$(d,2)$ symmetry manifest involves introducing a gauge symmetry, extra matter gauge degrees of freedom, and gauge fields.

B. Twistor space in d=4

The twistor formalism in $d=4$ [2][3] starts from a different description of the massless particle. Instead of phase space degrees of freedom x^{μ},p^{μ} that are SO$(3,1)$ vectors, it introduces SO$(3,1)$=SL$(2,C)$ spinor degrees of freedom $Z_A=\binom{\mu^{\dot\alpha}}{\lambda_\alpha}$, $A=1,2,3,4$, constructed from SL$(2,C)$ doublet spinors $\mu^{\dot\alpha}$, λ_α, each described by two complex degrees of freedom $\alpha,\dot\alpha=1,2$. The quartet Z_A is the spinor representation $\mathbf{4}$ of the conformal group SO$(4,2)$ =SU$(2,2)$, while its conjugate $\bar Z^A=\left(Z^{\dagger}C\right)^A=\left(\bar\lambda_{\dot\alpha}\ \bar\mu^{\alpha}\right)$,with the SU$(2,2)$ metric

$$C=\sigma_1\times 1, \tag{1.6}$$

is the anti-quartet $\bar{\mathbf{4}}$ that corresponds to the second spinor representation of SO$(4,2)$. An over-bar such as $\bar\lambda_{\dot\alpha}$ means complex conjugate of λ_α. The spinor is subject to a SU$(2,2)$ invariant helicity constraint $Z_A\bar Z^A=\mu^{\dot\alpha}\bar\lambda_{\dot\alpha}+\lambda_\alpha\bar\mu^{\alpha}=2h$, where h is the helicity of the particle. The helicity constraint is the generator of a U(1) gauge symmetry that acts on the twistor through the local phase transformation $Z_A(\tau)\rightarrow Z'_A(\tau)=e^{i\Lambda(\tau)}Z_A(\tau)$. The gauge invariant action that describes the dynamics of twistors in four dimensions is

$$S(Z)=\int d\tau\left(i\bar Z^A DZ_A-2hV\right),\quad DZ_A\equiv\frac{\partial Z_A}{\partial\tau}-iVZ_A. \tag{1.7}$$

Here the 1-form $Vd\tau$ is a U(1) gauge field on the worldline, DZ_A is the gauge covariant derivative that satisfies $\delta_\Lambda(DZ_A)=i\Lambda(\tau)(DZ_A)$ for $\delta_\Lambda V=\partial\Lambda/\partial\tau$ and $\delta_\Lambda Z_A=i\Lambda(\tau)Z_A$. Note that the term $2hV$ (absent in previous literature) is gauge invariant since it transforms as a total derivative under the gauge transformation. The reason for requiring the U(1) gauge symmetry is the fact that the overall phase of the Z_A is unphysical and drops out in the relation between phase space and twistors, as in Eq.(1.8). Furthermore, the equation of motion with respect to V imposes the constraint $Z_A\bar Z^A-2h=0$, which is interpreted as the helicity constraint. Taking into account that $\left(Z_A\bar Z^A-2h\right)$ is the generator of the U(1) gauge transformations, the meaning of the vanishing generator (or helicity constraint) is that only the U(1) gauge invariant sector of twistor space is physical.

To establish the equivalence between the massless spinless particle in Eqs.(1.1,1.2) in $d=4$, and twistors with vanishing helicity $h=0$, we must choose a U(1) gauge for Z_A and solve the constraint $Z_A\bar{Z}^A=\mu^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}}+\lambda_\alpha\bar{\mu}^\alpha=0$. First we count degrees of freedom. The twistor Z_A has 4 complex , or 8 real, degrees of freedom. Gauge fixing the U(1) symmetry and solving the helicity constraint removes 2 real degrees of freedom, leaving behind 6 real degrees of freedom, which is the same number as the phase space degrees of freedom $(\vec{x},\vec{p})$.

More explicitly, Penrose has provided the transformation between twistors and the phase space of spinless massless particles as follows

$$\mu^{\dot{\alpha}}=-ix^{\dot{\alpha}\beta}\lambda_\beta,\ \lambda_\alpha\bar{\lambda}_{\dot{\beta}}=p_{\alpha\dot{\beta}}, \tag{1.8}$$

where the 2×2 Hermitian matrices $x^{\dot{\alpha}\beta}$, $p_{\alpha\dot{\beta}}$ are expanded in terms of the Pauli matrices

$$x^{\dot{\alpha}\beta}\equiv\frac{1}{\sqrt{2}}x^\mu\left(\bar{\sigma}_\mu\right)^{\dot{\alpha}\beta},\ p_{\alpha\dot{\beta}}\equiv\frac{1}{\sqrt{2}}p^\mu\left(\sigma_\mu\right)_{\alpha\dot{\beta}};\ \sigma_\mu\equiv(1,\vec{\sigma}),\ \bar{\sigma}_\mu\equiv(-1,\vec{\sigma}). \tag{1.9}$$

Here λ_α can be gauge fixed by choosing a phase, and the helicity constraint is explicitly solved since

$$\bar{Z}^A Z_A=\begin{pmatrix}\bar{\lambda}_{\dot{\alpha}} & \bar{\mu}^\alpha\end{pmatrix}\begin{pmatrix}\mu^{\dot{\alpha}}\\ \lambda_\alpha\end{pmatrix}=\bar{\lambda}_{\dot{\alpha}}\mu^{\dot{\alpha}}+\bar{\mu}^\alpha\lambda_\alpha=-i\bar{\lambda}_{\dot{\alpha}}x^{\dot{\alpha}\beta}\lambda_\beta+i\bar{\lambda}_{\dot{\beta}}x^{\dot{\beta}\alpha}\lambda_\alpha=0. \tag{1.10}$$

Furthermore, since $\lambda_\alpha\bar{\lambda}_{\dot{\beta}}$ is a 2×2 matrix of rank one, its determinant vanishes. Then the parametrization $p_{\alpha\dot{\beta}}=\lambda_\alpha\bar{\lambda}_{\dot{\beta}}$ of the momentum p^μ, with $\sqrt{2}p^0=Tr\,(p)=\bar{\lambda}\lambda$, insures automatically that $p^0>0$ and that the mass shell condition is satisfied $\det\left(\lambda\bar{\lambda}\right)=\det(p)=p^\mu p_\mu=0$. Furthermore, by inserting the twistor transform in Eq.(1.8) into the twistor action $S(Z)$ we recover the massless particle action Eq.(1.2) on which $p^2=0$ is already imposed by the form of p^μ

$$S(Z)=\int d\tau\left[\bar{\lambda}_{\dot{\alpha}}\frac{\partial}{\partial\tau}\left(x^{\dot{\alpha}\beta}\lambda_\beta\right)-\bar{\lambda}_{\dot{\alpha}}x^{\dot{\alpha}\beta}\frac{\partial\lambda_\beta}{\partial\tau}+0\right]=\int d\tau\frac{\partial x^{\dot{\alpha}\beta}}{\partial\tau}\lambda_\beta\bar{\lambda}_{\dot{\alpha}} \tag{1.11}$$

$$=\frac{1}{2}\int d\tau\frac{\partial x_\mu}{\partial\tau}p_\nu Tr\left(\bar{\sigma}^\mu\sigma^\nu\right)=\int d\tau\frac{\partial x_\mu}{\partial\tau}p_\mu,\ (\text{with } p^2=0) \tag{1.12}$$

Finally, fixing the τ reparametrization symmetry $x^0=\tau$ shows that $S(Z)$ gives the original massless particle action Eq.(1.1). From this we conclude that the canonical structure of twistors determined by $S(Z)$, namely $\left[Z_A,\bar{Z}^B\right]=\delta_A^{\ B}$, is equivalent to the canonical structure in phase space determined by $S(x,p)$, namely $[x^\mu,p_\nu]=i\delta^\mu_{\ \nu}$, in the gauge invariant sectors determined respectively by the constraints $\bar{Z}Z=0$ and $p^2=0$.

The analog of covariant quantization described in Eq.(1.5) can also be done in twistor space. At the quantum level the Hermitian ordered product is applied on physical states $\frac{1}{2}(Z_A\bar{Z}^A+\bar{Z}^A Z_A)|\psi\rangle=2h|\psi\rangle$. Wavefunctions in twistor space are obtained as $\psi(Z)=\langle Z|\psi\rangle$ where the operators Z_A are diagonalized on the states labelled as $\langle Z|$. Using the fact that the

canonical conjugate acts as $\bar{Z}^A\psi(Z) = \langle Z|\bar{Z}^A|\psi\rangle = -\frac{\partial}{\partial Z_A}\psi(Z)$, the physical state condition in Z space $\frac{1}{2}\langle Z|(Z_A\bar{Z}^A + \bar{Z}^A Z_A)|\psi\rangle = 2h\langle Z|\psi\rangle$ produces Penrose's homogeneity constraint, $Z_A\frac{\partial}{\partial Z_A}\psi(Z) = (-2h-2)\psi(Z)$, known to correctly describe the quantum wavefunction $\psi(Z)$ of a particle of helicity h.

As the analog of the Klein-Gordon field theory of Eq.(1.5), we propose an action in twistor space field theory that takes into account at once both the positive and negative helicities

$$S_h(\psi) = \int d^4Z\ \psi^*\left(Z_A\frac{\partial}{\partial Z_A} + 2h + 2\right)\psi. \tag{1.13}$$

The minimal action principle yields the homogeneity constraints. Indeed, the equations of motion derived from $S_h(\psi)$ for ψ, ψ^* show that $\psi(Z)$ is the helicity $+h$ wavefunction, while $\psi^*(Z)$ is the helicity $-h$ wavefunction, so that together they describe a CPT invariant free field theory. Therefore $S_h(\psi)$ is the twistor equivalent of the Klein-Gordon, Dirac, Maxwell and higher spinning particle free field actions in four flat dimensions. The free field theory $S_h(\psi)$ is evidently invariant under conformal transformations $\mathrm{SU}(2,2)$ by taking $\psi(Z)$ to transform like a scalar while Z_A transforms like the fundamental representation of $\mathrm{SU}(2,2)$. Our field theory proposal for an action principle $S_h(\psi)$ for any spinning particle seems to be new in the literature.

What do we learn from the twistor approach? Perhaps, as Penrose would suggest, twistors may be more basic than spacetime? Without a conclusive answer to that question so far, it is nevertheless evident through the twistor program [2],[3], and the recent twistor superstring [8]-[13] that led to computational advances in Super Yang Mills theory [14][15], that twistor space is a useful space, as an alternative to space-time, to discuss the physics of massless systems.

But there is more to be said about twistors and spacetime. In addition to what has traditionally been known about twistors, it has recently been shown [7] that the *same* twistor Z_A in Eq.(1.7) which is known to describe the on-shell massless particle, also describes a variety of other on-shell dynamical systems. In particular the twistor transform of Eq.(1.8) has been generalized so that the *same* twistor also gives the d=4 particle worldline actions for the massive relativistic particle, the particle on AdS_4 or $\mathrm{AdS}_3\times\mathrm{S}^1$ or $\mathrm{AdS}_2\times\mathrm{S}^2$, the particle on $R\times S^3$, the nonrelativistic free particle in 3 space dimensions, the nonrelativistic hydrogen atom in 3 space dimensions, and a related family of other particle systems. For example the twistor transform for the massive relativistic particle is [7]

$$\mu^{\dot{\alpha}} = -ix^{\dot{\alpha}\beta}\lambda_\beta\frac{2a}{1+a},\ \ \lambda_\alpha\bar{\lambda}_{\dot{\beta}} = \frac{1+a}{2a}p_{\alpha\dot{\beta}} + \frac{m^2}{2(x\cdot p)a}x_{\alpha\dot{\beta}}, \tag{1.14}$$

where $a \equiv \sqrt{1+\frac{m^2x^2}{(x\cdot p)^2}}$. Using this transform instead of Eq.(1.8) we find that the action $S(Z)$ in Eq.(1.7) reduces to the action for the massive particle $S = \int dt\left(\partial_t\vec{x}\cdot\vec{p} - \sqrt{\vec{p}^2+m^2}\right)$

instead of the massless particle of Eq.(1.1). The mass parameter emerges as a modulus in relating the twistor components $\left(\mu^{\dot{\alpha}}, \lambda_{\alpha}\right)$ to phase space (x^{μ}, p^{μ}) in a different way than Eq.(1.8). As seen in [7] the mass parameter can also be thought of as the value of an extra momentum component in $4+2$ dimensions. Similarly in the case of the twistor transform for the H-atom, a combination of mass and the Coulomb coupling constant is a modulus, and so on for other moduli in more general cases. We see that certain mass parameters, certain curvature or other spacetime metric parameters, and certain coupling constants emerge as moduli in the generalized twistor transform.

The results in [7] imply that twistor space is a unifying space for various dynamics, with different Hamiltonians, which must be related to one another through a web of dualities. This raises deeper questions on the meaning of space and time, and accentuates the feeling that twistor space may be even more fundamental than was thought of before: namely, from the point of view of twistor space, spacetime and dynamics are emergent concepts, as explicitly shown in the examples in ref.[7]. There seems to be a deeper meaning for twistors in the context of unification that goes beyond the originally envisaged role for twistors[2].

Actually ref.[7] provides a connection between the more general twistor properties just outlined and the concept of 2T-physics. This relation will be explained through the top→down approach in the next section. Here we will briefly describe the relevant properties of 2T-physics that unify various particle dynamics in 1T-physics, and thus promote the notion of spacetime to a higher level.

C. 2T-physics

2T-physics can be viewed as a unification approach for one-time physics (1T-physics) systems through higher dimensions. It is distinctly different than Kaluza-Klein theory because there are no Kaluza-Klein towers of states, but instead there is a family of 1T systems with duality type relationships among them.

A particle interacting with various backgrounds in $(d-1)+1$ dimensions (e.g. electromagnetism, gravity, high spin fields, any potential, etc.), usually described in a worldline formalism in 1T-physics, can be equivalently described in 2T-physics. The 2T theory is in $d+2$ dimensions, but has enough gauge symmetry to compensate for the extra $1+1$ dimensions, so that the physical (gauge invariant) degrees of freedom are equivalent to those

[2] Promoting these results to the quantum level we must expect another remarkable result that the wave equations that follow from our proposed action $S_h(\psi)$ in Eq.(1.13) must also correctly describe all the other cases unified by the same twistor as given in [7]. This point will not be further discussed in this paper and will be taken up in a future publication.

encountered in 1T-physics.

One of the strikingly surprising aspects of 2T-physics is that a given $d+2$ dimensional 2T theory descends, through gauge fixing, down to a family of holographic 1T images in $(d-1)+1$ dimensions. Each image fully captures the gauge invariant physical content of a unique parent 2T theory, but from the point of view of 1T-physics each image appears as a different 1T-dynamical system. The members of such a family naturally must obey duality-type relationships among them and share many common properties. In particular they share the same overall global symmetry in $d+2$ dimensions that becomes hidden and non-linear when acting on the fewer $(d-1)+1$ dimensions in 1T-physics. Thus 2T-physics unifies many 1T systems into a family that corresponds to a given 2T-physics parent in $d+2$ dimensions.

The essential ingredient in 2T-physics is the basic gauge symmetry Sp(2,R) acting on phase space X^M, P_M in $d+2$ dimensions. The two timelike directions is not an input, but is one of the outputs of the $\mathrm{Sp}(2,R)$ gauge symmetry. A consequence of this gauge symmetry is that position and momentum become indistinguishable at any instant, so the symmetry is of fundamental significance. The transformation of X^M, P_M is generally a nonlinear map that can be explicitly given in the presence of background fields [16], but in the absence of backgrounds the transformation reduces to a linear doublet action of $\mathrm{Sp}(2,R)$ on $\left(X^M, P^M\right)$ for each M [4]. The physical phase space is the subspace that is gauge invariant under $\mathrm{Sp}(2,R)$. Since $\mathrm{Sp}(2,R)$ has 3 generators, to reach the physical space we must choose 3 gauges and solve 3 constraints. So, the gauge invariant subspace of $d+2$ dimensional phase space X^M, P_M is a phase space with six fewer degrees of freedom in $(d-1)$ *space* dimensions (x^i, p_i), $i=1,2,\cdots(d-1)$.

In some cases it is more convenient not to fully use the three $\mathrm{Sp}(2,R)$ gauge symmetry parameters and work with an intermediate space in $(d-1)+1$ dimensions (x^μ, p_μ), that includes time. This space can be further reduced to $d-1$ space dimensions (x^i, p_i) by a remaining one-parameter gauge symmetry.

There are many possible ways to embed the $(d-1)+1$ or $(d-1)$ phase space in $d+2$ phase space, and this is done by making Sp(2,R) gauge choices. In the resulting gauge fixed 1T system, time, Hamiltonian, and in general curved spacetime, are emergent concepts. The Hamiltonian, and therefore the dynamics as tracked by the emergent time, may look quite different in one gauge versus another gauge in terms of the remaining gauge fixed degrees of freedom. In this way, a unique 2T-physics action gives rise to many 1T-physics systems.

The general 2T theory for a particle moving in any background field has been constructed [16]. For a spinless particle it takes the form

$$S=\int d\tau\ \left(\dot{X}^M P_M-\frac{1}{2}A^{ij}Q_{ij}\left(X,P\right)\right), \tag{1.15}$$

where the symmetric $A^{ij}(\tau)$, $i,j=1,2$, is the $\mathrm{Sp}(2,R)$ gauge field, and the three $\mathrm{Sp}(2,R)$ generators $Q_{ij}(X(\tau),P(\tau))$, which generally depend on background fields that are functions of $(X(\tau),P(\tau))$, are required to form an $\mathrm{Sp}(2,R)$ algebra. The background fields must satisfy certain conditions to comply with the $\mathrm{Sp}(2,R)$ requirement. An infinite number of solutions to the requirement can be constructed [16]. So any 1T particle worldline theory, with any backgrounds, can be obtained as a gauge fixed version of some 2T particle worldline theory.

The 1T systems discussed in [7], and alluded to in connection with twistors above, are obtained by considering the simplest version of 2T-physics without any background fields. The 2T action for a "free" 2T particle is [4]

$$S_{2T}(X,P)=\frac{1}{2}\int d\tau\ D_\tau X_i^M X_j^N \eta_{MN}\varepsilon^{ij}=\int d\tau\ \left(\dot{X}^M P^N-\frac{1}{2}A^{ij}X_i^M X_j^N\right)\eta_{MN}. \tag{1.16}$$

Here $X_i^M=\left(X^M\ P^M\right)$, $i=1,2$, is a doublet under $\mathrm{Sp}(2,R)$ for every M, the structure $D_\tau X_i^M=\partial_\tau X_i^M-A_i{}^j X_j^M$ is the Sp(2,R) gauge covariant derivative, Sp(2,R) indices are raised and lowered with the antisymmetric $\mathrm{Sp}(2,R)$ metric ε^{ij}, and in the last expression an irrelevant total derivative $-(1/2)\,\partial_\tau(X\cdot P)$ is dropped from the action. This action describes a particle that obeys the $\mathrm{Sp}(2,R)$ gauge symmetry, so its momentum and position are locally indistinguishable due to the gauge symmetry. The $\left(X^M,P^M\right)$ satisfy the $\mathrm{Sp}(2,R)$ constraints

$$Q_{ij}=X_i\cdot X_j=0:\ X\cdot X=P\cdot P=X\cdot P=0, \tag{1.17}$$

that follow from the equations of motion for A^{ij}. The vanishing of the gauge symmetry generators $Q_{ij}=0$ implies that the physical phase space is the subspace that is $\mathrm{Sp}(2,R)$ gauge invariant. These constraints have non-trivial solutions only if the metric η_{MN} has two timelike dimensions. So when position and momentum are locally indistinguishable, to have a non-trivial system, two timelike dimensions are necessary as a consequence of the $\mathrm{Sp}(2,R)$ gauge symmetry.

Thus the $\left(X^M,P^M\right)$ in Eq.(1.16) are $\mathrm{SO}(d,2)$ vectors, labelled by $M=0',1',\mu$ or $M=\pm',\mu$, and $\mu=0,1,\cdots,(d-1)$ or $\mu=\pm,1,\cdots,(d-2)$, with lightcone type definitions of $X^{\pm'}=\frac{1}{\sqrt{2}}\left(X^{0'}\pm X^{1'}\right)$ and $X^{\pm}=\frac{1}{\sqrt{2}}\left(X^0\pm X^3\right)$. The $\mathrm{SO}(d,2)$ metric η^{MN} is given by

$$ds^2=dX^M dX^N\eta_{MN}=-2dX^{+'}dX^{-'}+dX^\mu dX^\nu\eta_{\mu\nu} \tag{1.18}$$

$$=-\left(dX^{0'}\right)^2+\left(dX^{1'}\right)^2-\left(dX^0\right)^2+\left(dX^1\right)^2+\left(dX_\perp\right)^2 \tag{1.19}$$

$$=-2dX^{+'}dX^{-'}-2dX^+dX^-+\left(dX_\perp\right)^2. \tag{1.20}$$

where the notation $X_\perp$ indicates $\mathrm{SO}(d-2)$ vectors. So the target phase space X^M,P_M is flat in $d+2$ dimension, and hence the system in Eq.(1.16) has an $\mathrm{SO}(d,2)$ global symmetry.

The conserved generators of SO$(d,2)$

$$L^{MN} = X^M P^N - X^N P^M,\ \partial_\tau L^{MN} = 0, \tag{1.21}$$

commute with the SO$(d,2)$ invariant Sp$(2,R)$ generators $X\cdot X,\ P\cdot P,\ X\cdot P$.

The Sp$(2,R)$ local symmetry can be gauge fixed by choosing three gauges and solving three constraints, but to keep some of the subgroups of SO$(d,2)$ as evident symmetries it is more convenient to choose two gauges and solve two constraints.

The SO$(d-1,1)$ covariant massless particle emerges if we choose the two gauges, $X^{+'}(\tau) = 1$ and $P^{+'}(\tau) = 0$, and solve the two constraints $X^2 = X\cdot P = 0$ to obtain the $(d-1)+1$ dimensional phase space (x^μ, p_μ) embedded in $(d+2)$ dimensions

$$X^M = \left(\overset{+'}{1},\ \overset{-'}{x^2/2}\ ,\ \overset{\mu}{x^\mu} \right), \tag{1.22}$$

$$P^M = (\ 0\ ,\ x\cdot p\ ,\ \ p^\mu)\,. \tag{1.23}$$

The remaining constraint, $P^2 = -2P^{+'}P^{-'} + P^\mu P_\mu = p^2 = 0$, which is the third Sp$(2,R)$ generator, remains to be imposed on the physical sector. In this gauge the 2T-physics action in Eq.(1.16) reduces to the covariant massless particle action in Eq.(1.2). Furthermore, the Sp$(2,R)$ gauge invariant $L^{MN} = X^M P^N - X^N P^M$ take the following nonlinear form

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu,\ L^{+'-'} = x\cdot p,\ L^{+'\mu} = p^\mu,\ L^{-'\mu} = \frac{x^2}{2}p^\mu - x^\mu x\cdot p. \tag{1.24}$$

These are recognized as the generators of SO$(d,2)$ conformal transformations of the $(d-1)+$ 1 dimensional phase space at the classical level. Thus the conformal symmetry of the massless system is now understood as the Lorentz symmetry in $d+2$ dimensions.

Having established the higher symmetrical version of the theory for the massless particle as in Eq.(1.16) we reach a deeper level of understanding of the symmetries as well as the presence of the $d+2$ nature of the underlying spacetime. Furthermore we learn that the higher symmetrical parent theory can be gauge fixed in many ways that produce not only the massless particle system Eq.(1.1) we started from, but also an assortment of other particle dynamical systems, as discussed before [4][5][7].

To emphasize this point we give also the massive relativistic particle gauge by fixing two gauges and solving the constraints $X^2 = X\cdot P = 0$ explicitly as follows

$$X^M = \left(\overset{+'}{\frac{1+a}{2a}},\ \overset{-'}{\frac{x^2 a}{1+a}}\ ,\ \overset{\mu}{x^\mu} \right),\ a \equiv \sqrt{1 + \frac{m^2 x^2}{(x\cdot p)^2}} \tag{1.25}$$

$$P^M = \left(\frac{-m^2}{2(x\cdot p)a},\ \ (x\cdot p)\,a\ ,\ \ p^\mu \right),\ P^2 = p^2 + m^2 = 0. \tag{1.26}$$

In this gauge the 2T action reduces to the relativistic massive particle action

$$S=\int d\tau\ \left(\dot{X}^{M}P^{N}-\frac{1}{2}A^{ij}X_{i}^{M}X_{j}^{N}\right)\eta_{MN}=\int d\tau\left(\dot{x}^{\mu}p_{\mu}-\frac{1}{2}A^{22}\left(p^{2}+m^{2}\right)\right). \tag{1.27}$$

A little recognized fact is that this action is invariant under SO$(d,2)$. This SO$(d,2)$ does not have the form of conformal transformations of Eq.(1.24), but is a deformed version of it, including the mass parameter. Its generators are obtained by inserting the massive particle gauge into the gauge invariant $L^{MN}=X^{M}P^{N}-X^{N}P^{M}$

$$L^{\mu\nu}=x^{\mu}p^{\nu}-x^{\nu}p^{\mu},\qquad L^{+'-'}=\left(x\cdot p\right)a, \tag{1.28}$$

$$L^{+'\mu}=\frac{1+a}{2a}p^{\mu}+\frac{m^{2}}{2\left(x\cdot p\right)a}x^{\mu} \tag{1.29}$$

$$L^{-'\mu}=\frac{x^{2}a}{1+a}p^{\mu}-\left(x\cdot p\right)ax^{\mu} \tag{1.30}$$

It can be checked explicitly that the massive particle action above is invariant under the SO$(d,2)$ transformations generated by the Poisson brackets $\delta x^{\mu}=\frac{1}{2}\omega_{MN}\left\{L^{MN},x^{\mu}\right\}$ and $\delta p^{\mu}=\frac{1}{2}\omega_{MN}\left\{L^{MN},p^{\mu}\right\}$, up to a reparametrization of A^{22} by a scale and an irrelevant total derivative.

Since both the massive and massless particles give bases for the same representation of SO$(d,2)$, we must expect a duality transformation between them. Of course this transformation must be an Sp$(2,R)$ =SL$(2,R)$ local gauge transformation $\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)(\tau)$ with unit determinant $\alpha\delta-\beta\gamma=1$, that transforms the doublets $\left(\begin{smallmatrix}X^{M}\\ P^{M}\end{smallmatrix}\right)(\tau)$ from Eqs.(1.25,1.26) to Eqs.(1.22,1.23). The $\alpha,\beta,\gamma,\delta$ are fixed by focussing on the doublets labelled by $M=+'$

$$\begin{pmatrix}\left(\frac{1+a}{2a}\right)\\ \left(\frac{-m^{2}}{2(x\cdot p)a}\right)\end{pmatrix}=\begin{pmatrix}\left(\frac{1+a}{2a}\right)&0\\ \left(\frac{-m^{2}}{2(x\cdot p)a}\right)&\left(\frac{2a}{1+a}\right)\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}. \tag{1.31}$$

Applying the inverse of this transformation on the doublets labelled by $M=\mu$ gives the massless particle phase space (re-labelled by $(\tilde{x}^{\mu},\tilde{p}^{\mu})$ below) in terms of the massive particle phase space (labelled by (x^{μ},p^{μ}))

$$\begin{pmatrix}\left(\frac{2a}{1+a}\right)&0\\ \left(\frac{m^{2}}{2(x\cdot p)a}\right)&\left(\frac{1+a}{2a}\right)\end{pmatrix}\begin{pmatrix}x^{\mu}\\ p^{\mu}\end{pmatrix}=\begin{pmatrix}\frac{2a}{1+a}x^{\mu}\\ \frac{1+a}{2a}p^{\mu}+\frac{m^{2}}{2(x\cdot p)a}x^{\mu}\end{pmatrix}\equiv\begin{pmatrix}\tilde{x}^{\mu}\\ \tilde{p}^{\mu}\end{pmatrix} \tag{1.32}$$

This duality transformation is a canonical transformation $\left\{\tilde{x}^{\mu},\tilde{p}^{\nu}\right\}=\eta^{\mu\nu}=\left\{x^{\mu},p^{\nu}\right\}$. Also note that the time coordinate $\tilde{x}^{0}$ is different than the time coordinate x^{0}, and so are the corresponding Hamiltonians for the massless particle $\tilde{p}^{0}=\sqrt{\tilde{p}^{i}\tilde{p}^{i}}$ versus the massive particle $p^{0}=\sqrt{p^{i}p^{i}+m^{2}}$.

The same reasoning applies among all gauge choices of the 2T theory in Eq.(1.16). All resulting 1T dynamical systems are holographic images of the same parent theory. The global

symmetry SO$(d,2)$ of the 2T-physics action is shared in the same singleton[3] representation by all the emergent lower dimensional theories obtained by different forms of gauge fixing. These include special cases of particles that are massive or massless, relativistic and nonrelativistic, in flat or curved spaces, free or interacting. This is an established fact in previous work on 2T-physics [4][5], and it came into new focus by displaying the explicit twistor/phase space transforms given in [7].

As seen above, the descendants of the $d+2$ dimensional 2T-physics action are 1T-physics dynamical systems that are dual to each other. Therefore we must expect that they all have the same twistor representation modulo twistor gauge transformations. This will be derived through the top→down approach in the next section.

It must be emphasized that as a by product of the top→down approach certain physical parameters, such as mass, parameters of spacetime metric, and some coupling constants appear as moduli in the holographic image while descending from $d+2$ dimensional phase space to $(d-1)+1$ dimensions or to twistors. Explicit examples of these have appeared in [7].

II. TOP→DOWN APPROACH

The 2T-physics action (1.16) and the twistor action (1.7) in four dimensions are related to one another and can both be obtained as gauge choices from the same theory in the 2T-physics formalism. To demonstrate this fact and setup a general formalism for deriving the twistor transform in any dimension, with or without supersymmetry, we discuss a unified theory that defines the top→down approach. This formalism was introduced in [6] and developed further in the context of the twistor superstring [12][13]. In this section we begin without supersymmetry or compactified dimensions. These will be introduced later.

In the case of $d=4$ the generalized twistor transform was applied explicitly to specific cases in [7], but the derivation of the general formula was relegated to the present paper. In this section we will derive the general twistor transform between twistor space Z in d-dimensions and the $d+2$ dimensional phase space X^M, P_M or d dimensional phase space x^μ, p_μ. We will show how it works explicitly in $d=3,4,5,6$ and higher dimensions.

[3] At the classical level all Casimir eigenvalues vanish, but at the quantum level, due to ordering of factors the Casimir eigenvalues are non-zero (see Eq.(2.9)) and correspond to the singleton representation.

A. SO$(d,2)$ local and SO$(d,2)$ global symmetry

In addition to the phase space SO$(d,2)$ vectors $\left(X^M, P^M\right)$, we introduce a group element $g(\tau) \in$SO$(d,2)$ in the *spinor* representation. It is given by

$$g(\tau) = \exp\left(\frac{i}{2} S^{MN} \omega_{MN}(\tau)\right) = \exp\left(\frac{1}{4}\Gamma^{MN}\omega_{MN}(\tau)\right) \tag{2.1}$$

The 2T particle action with Sp$(2,R)$ & SO$(d,2)$ local and SO$(d,2)$ global symmetry is

$$S_{2T}(X,P,g) = \int d\tau \left[\frac{1}{2}\varepsilon^{ij}\partial_\tau X_i \cdot X_j - \frac{1}{2}A^{ij}X_i \cdot X_j + \frac{4}{s_d} Tr\left(ig^{-1}\partial_\tau g L\right)\right], \tag{2.2}$$

where the trace is in spinor space[4] and the matrix L is given by

$$L \equiv \frac{1}{4i}\Gamma_{MN}L^{MN} = \frac{1}{4i}\left(\Gamma\cdot X\ \bar\Gamma\cdot P - \Gamma\cdot P\ \bar\Gamma\cdot X\right). \tag{2.3}$$

The first two terms of the action $S_{2T}(X,P,g)$ are the same as Eq.(1.16), hence these terms are invariant under Sp$(2,R)$ which acts on $X_i^M = \left(X^M, P^M\right)$ as a doublet for every M, and on A^{ij} as the gauge field. Furthermore, by taking $g(\tau)$ as a singlet while noting that $L^{MN} = \varepsilon^{ij}X_i^M X_j^N = X^M P^N - X^N P^M$ is Sp$(2,R)$ gauge invariant, we see that the full action is gauge invariant under Sp$(2,R)$. The action can be rewritten in the form

$$S_{2T}(X,P,g) = \int d\tau \left\{\frac{1}{2s_d}\varepsilon^{ij} Tr\left[\partial_\tau\left(gX_i\cdot\Gamma g^{-1}\right)\left(gX_j\cdot\bar\Gamma g^{-1}\right)\right] - \frac{1}{2}A^{ij}X_i\cdot X_j\right\}. \tag{2.4}$$

When both X_i^M and $g(\tau)$ are transformed under local Lorentz transformations as $\delta_R X_i^M = \varepsilon_R^{MN}(\tau) X_{iN}$ and $\delta_R g = -\frac{1}{4}\left(g\Gamma_{MN}\right)\varepsilon_R^{MN}(\tau)$, the structures $(gX_j\cdot\Gamma g^{-1})$ and $X_i{\cdot}X_j$ are gauge invariant under δ_R. Therefore the Lagrangian has a gauge symmetry with local SO$(d,2)_R$ parameters $\varepsilon_R^{MN}(\tau)$ when g is transformed on the right side. In addition, there is a global symmetry under SO$(d,2)_L$ when $g(\tau)$ is transformed from the left side as $\delta_L g = \frac{1}{4}\varepsilon_L^{MN}\left(\Gamma_{MN} g\right)$, with τ independent parameters ε_L^{MN}.

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[4] The trace in spinor space gives the dimension of the spinor $Tr(1) = s_d$ and $Tr\left(\Gamma^M\bar\Gamma^N\right) = s_d\eta^{MN}$. For even dimensions $s_d = 2^{d/2}$ for the Weyl spinor of SO$(d,2)$, and the $\bar\Gamma^M, \Gamma^M$ are the gamma matrices in the bases of the two different spinor representations.The correctly normalized generators of SO$(d,2)$ in the spinor representation are $S^{MN} = \frac{1}{2i}\Gamma^{MN}$, where the even-dimension gamma matrices satisfy $\Gamma^M\bar\Gamma^N + \Gamma^N\bar\Gamma^M = 2\eta^{MN}$, while $\Gamma^{MN} = \frac{1}{2}\left(\Gamma^M\bar\Gamma^N - \Gamma^N\bar\Gamma^M\right)$, $\Gamma^{MNK} = \frac{1}{3!}\left(\Gamma^M\bar\Gamma^N\Gamma^K \mp \text{permutations}\right)$, etc. There exists a metric C of SO$(d,2)$ in the spinor representation such that when combined with hermitian conjugation it gives $C^{-1}\left(\Gamma^M\right)^\dagger C = -\bar\Gamma^M$ and $C^{-1}\left(\Gamma^{MN}\right)^\dagger C = -\Gamma^{MN}$. So the inverse g^{-1} is obtained by combining hermitian and C-conjugation $g^{-1} = C^{-1}(g)^\dagger C \equiv \bar g$. In odd number of dimensions the even-dimension gamma matrices above are combined to a larger matrix $\hat\Gamma^M = \begin{pmatrix} 0 & \bar\Gamma^M \\ \Gamma^M & 0\end{pmatrix}$ for $M = 0', 1', 0, 1, \cdots, (d-2)$ and add one more matrix for the additional last dimension $\hat\Gamma^{d-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$. The text is written as if d is even; for odd dimensions we replace everywhere $\hat\Gamma^M$ for both Γ^M and $\bar\Gamma^M$.

Using Noether's theorem we construct the conserved charge $J^{MN}(\tau)$ of the global left side symmetry $\mathrm{SO}(d,2)_L$. We find $J^{MN} \sim iTr\left(\Gamma^{MN} gLg^{-1}\right)$, but we prefer to write it in spinor space in the form

$$J_A^{\ B} = \left(gLg^{-1}\right)_A^{\ B} = J^{MN}(\tau)\left(\frac{1}{4i}\Gamma_{MN}\right)_A^{\ B}. \tag{2.5}$$

Note that the matrix $J_A^{\ B}$ must have the same form as the matrix $L_A^{\ B}$ of Eq.(2.3), i.e. $J = \left(\frac{1}{4i}\Gamma_{MN}\right) J^{MN}(\tau)$, since $gLg^{-1} = \frac{1}{4i}\left(g\Gamma_{MN}g^{-1}\right) L^{MN}$ is a Lorentz transformation of the gamma matrices that mixes them among themselves. By using the equations of motion for (X, P, g) one can show that these charges are conserved $\partial_\tau J_A^{\ B}(\tau) = 0$.

As seen from the form of J in Eq.(2.5), it is gauge invariant under the local $\mathrm{SO}(d,2)_R$ as well as the local $\mathrm{Sp}(2,R)$ transformations. Therefore the $\mathrm{SO}(d,2)_L$ charges $J_A^{\ B}$ are physical observables that classify the physical states under $\mathrm{SO}(d,2)_L$ representations. In particular the Casimir operators of these representations are given by $C_n = \frac{1}{s_d} tr\left((2J)^n\right)$. With this in mind we study the properties of J. In particular the square of the matrix J, given by $\left(J^2\right)_A^{\ B} = \left(gLg^{-1}gLg^{-1}\right)_A^{\ B} = \left(gL^2g^{-1}\right)_A^{\ B}$, contains important information about the physical states as we will see below. To proceed from here we will outline the rest of the computation of J^2 at the classical and quantum levels.

If the square of the matrix L^2 is computed at the classical level, i.e. not caring about the orders of generators L_{MN}, then one finds that $\left(L^2\right)_A^{\ B}$ is proportional to the identity matrix δ_A^B, $\left(L^2\right) = \left(\frac{1}{4i}\Gamma_{MN}L^{MN}\right)^2 = \frac{1}{8}L^{MN}L_{MN}\ 1$. Furthermore by computing, still at the classical level $\frac{1}{2}L^{MN}L_{MN} = X^2P^2 - (X \cdot P)^2$, and imposing the classical constraints $X^2 = P^2 = (X \cdot P) = 0$, one finds that $L^2 = 0$ in the space of gauge invariants of the classical theory. Then this implies also $J^2 = 0$ in the space of gauge invariants of the classical theory. By taking higher powers of J, we find $J^n = 0$ for all positive integers $n \geq 2$. Therefore all Casimir eigenvalues are zero $C_n = 0$ for all the classical physical configurations of phase space. This is a special non-trivial representation of the non-compact group $\mathrm{SO}(d,2)_L$, and all classical gauge invariants, which are functions of L^{MN}, can be classified as irreducible multiplets of $\mathrm{SO}(d,2)_L$.

We now consider the quantum theory. All the physical (gauge invariant) states must fall into irreducible representations of the global symmetry $\mathrm{SO}(d,2)_L$. In the quantum theory the L_{MN} form the Lie algebra of $\mathrm{SO}(d,2)$, therefore if the square of the matrix L is computed at the quantum level, by taking into account the orders of the operators L^{MN}, one finds

$$L^2 = \left(\frac{1}{4i}\Gamma_{MN}L^{MN}\right)^2 = -\frac{d}{2}\left(\frac{1}{4i}\Gamma_{MN}L^{MN}\right) + \frac{1}{8}L^{MN}L_{MN}\ 1. \tag{2.6}$$

In this computation we used the properties of gamma matrices

$$\Gamma_{MN}\Gamma_{RS} = \Gamma_{MNRS} + \left(\eta_{NR}\Gamma_{MS} - \eta_{MR}\Gamma_{NS} - \eta_{NS}\Gamma_{MR} + \eta_{MS}\Gamma_{NR}\right) + \left(\eta_{NR}\eta_{MS} - \eta_{MR}\eta_{NS}\right).$$

The term $\Gamma_{MNRS}L^{MN}L^{RS}$ vanishes for $L^{MN} = X^{[M}P^{N]}$ due to a clash between symmetry/antisymmetry. The term "$\eta_{NR}\Gamma_{MS}\cdots$" turns into a commutator, and after using the SO$(d,2)$ Lie algebra for $\left[L^{MN}, L^{RS}\right]$ it produces the linear term proportional to $d/2$ in Eq.(2.6). The term "$\eta_{NR}\eta_{MS}\cdots$" produces the last term in Eq.(2.6). Furthermore the Casimir $\frac{1}{2}L^{MN}L_{MN}$ does not vanish at the quantum level. As shown in [4], in the Sp$(2,R)$ gauge invariant physical sector of phase space one finds that it has the fixed value $\frac{1}{2}L^{MN}L_{MN} = 1-d^2/4$ rather than zero. Hence, in the physical sector of the quantum theory the matrix $J_A^{\ B}$ satisfies the following algebra

$$\left(J^2\right)_A^{\ B} = -\frac{d}{2}\ J_A^{\ B} + \frac{1}{8}\left(1-\frac{d^2}{4}\right)\ \delta_A^{\ B},\ \text{on physical states.} \tag{2.7}$$

We can compute the higher powers J^n on physical states by repeatedly using this relation,

$$\left(J^n\right)_A^{\ B} = \alpha_n\ J_A^{\ B} + \beta_n\ \delta_A^{\ B}, \tag{2.8}$$

and then compute the Casimir eigenvalues[5] $C_n = \frac{1}{s_d}Tr\left(\left(2J\right)^n\right) = 2^n\beta_n$. Evidently the C_n will end up having fixed values determined by the dimension d of SO$(d,2)_R$. In particular,

$$C_2 = 1-\frac{d^2}{4},\ \ C_3 = d\left(1-\frac{d^2}{4}\right),\ C_4 = \left(1-\frac{d^2}{4}\right)\left(1+\frac{3d^2}{4}\right),\ \text{etc.} \tag{2.9}$$

Therefore, at the quantum level we have identified a special unitary representation that classifies all physical states of the theory. This is the singleton representation of SO$(d,2)$ for any d. Our approach shows that the singleton is more fully characterized by the constraints satisfied by the charges in Eq.(2.7). We will see in the next section that these constraints will be satisfied explicitly at the quantum level by constructing $J_A^{\ B}$ in terms of twistors.

B. Twistor gauge and the general twistor transform

There are different ways of choosing gauges to express the theory given by $S_{2T}\left(X,P,g\right)$ in terms of the physical sector. One extreme in gauge space is to eliminate g completely, while another extreme is to eliminate (X,P) completely. When g is eliminated we obtain the phase space description, and when (X,P) is eliminated we obtain the twistor description.

Since SO$(d,2)_R$ is a local symmetry that acts on $g\left(\tau\right)$ from the right, $g\rightarrow g'=gg_R$, and g_R has exactly the same number of degrees of freedom as g, one can gauge fix the extended classical theory by choosing the gauge $g\left(\tau\right)=1$. In that case the theory described by Eq.(2.3) in terms of (X,P,g) reduces to the theory described by only (X,P) in Eq.(1.16)

$$S_{2T}\left(X,P,g\right) \overset{g=1}{=} S_{2T}\left(X,P\right),\ \ J_A^{\ B} = L_A^{\ B}. \tag{2.10}$$

[5] Note that in the literature one may find that the definition of the cubic and higher Casimir eigenvalues are given as a linear combination of our C_n.

In this gauge the conserved charge becomes $J = L$, so that the global symmetry $\mathrm{SO}(d,2)_L$ becomes the $\mathrm{SO}(d,2)$ global symmetry of the (X,P) theory. This reflects the fact that to maintain the gauge $g = 1$, a transformation of g from the left must be compensated by a transformation from the right, therefore $\mathrm{SO}(d,2)_L$ and $\mathrm{SO}(d,2)_R$ become identified.

In the $g = 1$ gauge there still remains the $\mathrm{Sp}(2,R)$ gauge symmetry. If one fixes this gauge as in Eqs.(1.22,1.23) then we see that the original $\mathrm{SO}(d,2)_L$ is interpreted, in this gauge, as the conformal symmetry of the relativistic massless particle given in Eqs.(1.24). But if one fixes $\mathrm{Sp}(2,R)$ as in Eqs.(1.25-1.26) then the original $\mathrm{SO}(d,2)_L$ is interpreted as the hidden $\mathrm{SO}(d,2)$ of the massive particle given in Eqs.(1.28-1.30). So, the original $\mathrm{SO}(d,2)_L$ applied on g can take on many possible physical interpretations as the hidden symmetry of various dynamical particle phase spaces that arise from $\mathrm{Sp}(2,R)$ gauge choices. Recall that for all cases the conserved $\mathrm{SO}(d,2)$ charges are just the physical charges $J_A^{\ B} = \left(g^{-1}Lg\right)_A^{\ B}$ whose classical and quantum properties were already computed in a gauge invariant way in the previous section.

To obtain the twistor description of the system we eliminate $\left(X^M, P^M\right)$ completely and keep only g as discussed in [6]. This is done by using the $\mathrm{Sp}(2,R)$ and the $\mathrm{SO}(d,2)_R$ local symmetries to completely fix X^M, P^M to the convenient form $X^{+'} = 1$ and $P^{+} = 1$, while all other components vanish

$$X^M = (\overset{+'}{1}, \overset{-'}{0}, \overset{+}{0}, \overset{-}{0}, \overset{i}{0}), \; P^M = (\overset{+'}{0}, \overset{-'}{0}, \overset{+}{1}, \overset{-}{0}, \overset{i}{0}), \; i = 1, \cdots, (d-2)\,. \tag{2.11}$$

These X^M, P^M already satisfy the constraints $X^2 = P^2 = X \cdot P = 0$. In this gauge the only non-vanishing component of L^{MN} is $L^{+'+} = 1$, so that

$$L_{fixed} = \frac{-2}{4i}\Gamma^{-'-} L^{+'+} = \frac{i}{2}\Gamma^{-'-} \equiv \Gamma. \tag{2.12}$$

Hence the physical content of the theory is now described only in terms of g and the fixed matrix Γ embedded in the Lie algebra of $\mathrm{SO}(d,2)$.

The matrix Γ has very few non-zero entries as seen by choosing a convenient form of gamma matrices[6] for $\mathrm{SO}(d,2)$. Then, up to similarity transformations, Γ can be brought to

[6] An explicit form of $\mathrm{SO}(d,2)$ gamma matrices that we find convenient in even dimensions, is given by $\Gamma^0 = 1 \times 1$, $\Gamma^i = \sigma_3 \times \gamma^i$, $\Gamma^{\pm'} = -i\sqrt{2}\sigma^{\pm} \times 1$ (note $\Gamma^{0'} = -i\sigma_1 \times 1$ and $\Gamma^{1'} = \sigma_2 \times 1$), where γ^i are the $\mathrm{SO}(d-1)$ gamma matrices. The $\bar{\Gamma}^M$ are the same as the Γ^M for $M = \pm', i$, but for $M = 0$ we have $\bar{\Gamma}^0 = -\Gamma^0 = -1 \times 1$. From these we construct the traceless $\Gamma^{+'-'} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\Gamma^{+'\mu} = i\sqrt{2}\left(\begin{smallmatrix} 0 & \bar{\gamma}^{\mu} \\ 0 & 0 \end{smallmatrix}\right)$, $\Gamma^{-'\mu} = -i\sqrt{2}\left(\begin{smallmatrix} 0 & 0 \\ \gamma^{\mu} & 0 \end{smallmatrix}\right)$, $\Gamma^{\mu\nu} = \left(\begin{smallmatrix} \bar{\gamma}^{\mu\nu} & 0 \\ 0 & \gamma^{\mu\nu} \end{smallmatrix}\right)$, with $\gamma_\mu = \left(1, \gamma^i\right)$ and $\bar{\gamma}_\mu = \left(-1, \gamma^i\right)$. Then $\frac{1}{2}\Gamma_{MN}J^{MN} = -\Gamma^{+'-'}J^{+'-'} + \frac{1}{2}J_{\mu\nu}\Gamma^{\mu\nu} - \Gamma^{+'}_{\ \mu}J^{-'\mu} - \Gamma^{-'}_{\ \mu}J^{+'\mu}$ takes the matrix form given in Eq.(2.19). We can further write $\gamma^1 = \tau^1 \times 1$, $\gamma^2 = \tau^2 \times 1$ and $\gamma^r = \tau^3 \times \rho^r$, where the ρ^r are the gamma matrices for $\mathrm{SO}(d-3)$. These gamma matrices are consistent with the metric $C = \sigma_1 \times 1 \times c$ of Eq.(1.6), and footnote (4), provided

the form[7]

$$\Gamma=\frac{i}{2}\Gamma^{-'-}=\frac{1}{\sqrt{2}}\begin{pmatrix}0&0\\ \gamma^{-}&0\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}. \tag{2.13}$$

The identity matrix 1, and the 0's, are $\frac{s_d}{4}\times\frac{s_d}{4}$ square block matrices embedded in the spinor representation of $\mathrm{SO}(d,2)$. Then the gauge invariant 2T action in Eq.(2.2), and the gauge invariant $\mathrm{SO}(d,2)_L$ charges in Eq.(2.5), take the twistor form similar to Eq.(1.7)

$$S_{2T}(X,P,g)=\frac{4}{s_d}\int d\tau\ Tr\left(i\partial_\tau g\Gamma g^{-1}\right)=\frac{4}{s_d}\int d\tau\ i\partial_\tau Z_A^{\ a}\bar{Z}_a^{\ A}\equiv S_{twistor}, \tag{2.14}$$

$$(J)_A^B=\left(g\Gamma g^{-1}\right)_A^B=\left(Z_A^{\ a}\bar{Z}_a^{\ B}-\frac{1}{s_d}tr\left(Z\bar{Z}\right)\delta_A^B\right), \tag{2.15}$$

The $Z_A^{\ a},\bar{Z}_a^{\ B}$ are the twistors that already obey the constraints $\left(\bar{Z}Z\right)_a^{\ b}=0$ in Eq.(2.16) below, so $\frac{4}{s_d}\int d\tau\ i\partial_\tau Z_A^{\ a}\bar{Z}_a^{\ A}$ is the full twistor action (for an equivalent gauge invariant action that also produces the constraints, see Eq.(2.40)). Due to the form of Γ it is useful to think of g as written in the form of $\frac{s_d}{4}\times\frac{s_d}{4}$ square blocks. Then $Z_A^{\ a}$ with $A=1,2,\cdots,s_d$ and $a=1,2,\cdots,\frac{s_d}{4}$ emerges as the rectangular matrix that corresponds to the last block of columns of the matrix g, and similarly $\bar{Z}_a^{\ A}$ corresponds to the second block of rows of g^{-1}. Since $g^{-1}=C^{-1}g^{\dagger}C$, we find that $\bar{Z}=c^{-1}Z^{\dagger}C$, where $C=\sigma_1\times 1\times c$ is given in footnote (6). Furthermore, as part of g,g^{-1}, the $Z_A^{\ a},\bar{Z}_a^{\ B}$ must satisfy the constraint $\bar{Z}_a^{\ A}Z_A^{\ b}=0$ since the product $\bar{Z}_a^{\ A}Z_A^{\ b}$ contributes to an off-diagonal block of the matrix 1 in $g^{-1}g=1$,

$$g^{-1}g=1\ \rightarrow \bar{Z}_a^{\ A}Z_A^{\ b}=0. \tag{2.16}$$

A constraint such as this one must be viewed as the generator of a gauge symmetry that operates on the a index (the columns) of the twistor $Z_A^{\ a}$.

Let us do some counting of degrees of freedom. To describe the particle in d dimensions we only need $2(d-1)$ physical degrees of freedom corresponding to phase space $(\vec{x},\vec{p})$. This counting applies no matter if the particle is massless or massive, relativistic or not relativistic, in flat space or curved space, interacting or not interacting. Our twistors are expected to apply to all these cases, so we must have the same number of physical parameters in the twistors given above. Any extra parameters in $Z_A^{\ a}$ beyond $2(d-1)$ must be either gauge degrees of freedom of the twistor, or there must be additional relations among the $Z_A^{\ a}$. The $s_d\times s_d$ matrix g (with $s_d=2^{d/2}$ for even d) is constructed from $\frac{1}{2}(d+2)(d+1)$ group

$c^{-1}(\rho^r)^{\dagger}c=\rho^r$. It is possible to choose hermitian ρ^r with $c=1$ for $\mathrm{SO}(d-3)$. If one works in a basis with $c\neq 1$, then hermitian conjugation of of $\mathrm{SO}(d-3)$ spinors (which occur e.g. in $\bar{\lambda}$ of Eq.(2.18)) must be supplemented by multiplying with c, as in $\bar{\lambda}\equiv\lambda^{\dagger}(1\times c)$.

[7] The gamma matrices Γ^M of footnote (6) can be redefined differently for the left or right sides of g up to similarity transformations. Thus, for the right side of g we apply a similarity transformation so that $\gamma^1=\tau^3\times 1$, etc., to obtain $\gamma^-=\left(\gamma^0-\gamma^1\right)/\sqrt{2}$ in the form given in Eq.(2.13).

parameters ω_{MN} as in Eq.(2.1), but only $2\,(d-1)$ of those parameters, corresponding to a coset, contribute in Eqs.(2.14,2.15) due to the form of Γ as will be explained in section (II C). So for sufficiently large d, we expect to find many relations among the $s_d^2/4 = 2^{2d-2}$ entries in the rectangular matrix $Z_A^{\ a}\,(g)$.

In $d = 3$, with $s_3/4 = 1$ columns, the Z_A form the fundamental representation of $\mathrm{Sp}(4,R)$ $=\mathrm{SO}(3,2)$. The single Z_A has just 4 real components that automatically satisfy the constraint $\bar{Z}^{\ A} Z_A = 0$. This number of degrees of freedom precisely matches the expected number 4 of physical degrees of freedom, $2\,(d-1) = 2\,(3-1) = 4$, for $d = 3$. So there are no extra relations among the 4 real twistor entries in Z_A.

In $d = 4$ dimensions, with $\mathrm{s}_4/4 = 1$ columns, the Z_A form the fundamental representation of $\mathrm{SU}(2,2)$ $=\mathrm{SO}(4,2)$. The single Z_A has 4 complex components or 8 real parameters that must satisfy the U(1) gauge constraint $\bar{Z}^{\ A} Z_A = 0$ of Eq.(2.16). The U(1) gauge symmetry together with the constraint remove 2 real parameters. So the Z_A contains $8-2 = 6$ physical degrees of freedom, which is just the correct number $2\,(4-1) = 6$ in $d = 4$, as discussed in section (I B). So there are no extra relations among the 4 complex twistor entries in Z_A.

For $d = 5, 6$, with $s_5/4 = s_6/4 = 2$ columns, the $Z_A^{\ a}$ is a doublet under an SU(2) gauge symmetry for $d = 6$, and $\mathrm{SU}(2)\times\mathrm{U}(1)$ for $d = 5$. Beyond the SU(2) or $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge freedom there seems to be further relations, but these amount to a simple pseudo-reality condition on $Z_A^{\ a}$, consistent with the transformation rules of $Z_A^{\ a}$ under $\mathrm{Spin}(6,2)\times\mathrm{SU}(2)$. The pseudo-reality condition emerges from the pseudo-reality of the spinor representation of $\mathrm{SO}(6,2)$ or $\mathrm{SO}(5,2)$. So, again there are no complicated relations among the entries of $Z_A^{\ a}$ for $d = 5, 6$ as seen by the following simple counting. The number of real entries in the pseudo-real $Z_A^{\ a}$ is $s_6^2/8 = s_5^2/8 = 8^2/4 = 16$. The 3 SU(2) gauge conditions together with the 3 constraints remove 6 parameters, leaving $16 - 6 = 10$, which is precisely the correct number of physical degrees of freedom for $d = 6$, i.e. $2\,(6-1) = 10$. Similarly, for $d = 5$ the extra U(1) and its constraint removes two more real parameters and this matches the correct number $2\,(5-1) = 8$.

For higher dimensions there are gauge symmetries among the columns but there also are further complicated relations among the $Z_A^{\ a}$. It turns out that in all cases $d \geq 3$ just the first column of $Z_A^{\ a}$ (i.e. $a = 1$) already contains all the parameters that describe the physical degrees of freedom, but it is still useful to deal with the full $Z_A^{\ a}$ since all components can be conveniently given explicitly in terms of gamma matrices, as seen in Eqs.(2.37,2.38) below.

We can construct explicitly the $Z_A^{\ a}, \bar{Z}_a^{\ A}$ that satisfy all of the relations discussed above at the classical level. This will give the twistor transform we are after. The key is the gauge invariant J_A^B. We identify its two different forms in the two different gauges, one in terms of

phase space and the other in terms of twistors, as given in Eqs.(2.10,2.15)

$$\frac{1}{4i}\Gamma_{MN}L^{MN} \overset{g=1}{=} J \overset{X,P\sim 0}{=} Z\bar{Z}, \text{ with } \left(\bar{Z}Z\right)_a^{\ b} = 0,\ Tr\left(Z\bar{Z}\right) = 0. \tag{2.17}$$

Of course, the $(X, P, g = 1)$ on the left side of the equation are gauge transformations of the $(X^M = \delta^M_{+'},\ P^M = \frac{s_d}{4}\delta^M_{+},$ and $Z(g))$ on the right side. So this equation must contain the twistor transform. More explicitly we write Z in terms of its components

$$Z_A^{\ a} = \begin{pmatrix} \mu \\ \lambda \end{pmatrix},\ \bar{Z}_a^{\ A} = \begin{pmatrix} \bar{\lambda} & \bar{\mu} \end{pmatrix},\ \bar{Z}_a^{\ A} Z_A^{\ b} = \left(\bar{\lambda}\mu + \bar{\mu}\lambda\right)_a^{\ b} = 0, \tag{2.18}$$

where μ, λ are $\frac{s_d}{2} \times \frac{s_d}{4}$ rectangular matrices. Then we use the gamma matrices, and the definition of $\bar{\lambda}, \bar{\mu}$ in footnote (6), to write the basic relation (2.17) between phase space and twistors more explicitly as

$$\frac{1}{2i}\begin{pmatrix} L^{+'-'} + \frac{1}{2}L_{\mu\nu}\bar{\gamma}^{\mu\nu} & -i\sqrt{2}L^{-'\mu}\ \bar{\gamma}_\mu \\ i\sqrt{2}L^{+'\mu}\ \gamma_\mu & -L^{+'-'} + \frac{1}{2}L_{\mu\nu}\gamma^{\mu\nu} \end{pmatrix} = J = \begin{pmatrix} \mu\bar{\lambda} & \mu\bar{\mu} \\ \lambda\bar{\lambda} & \lambda\bar{\mu} \end{pmatrix}. \tag{2.19}$$

Comparing the lower off diagonal blocks we learn part of the twistor transform

$$\lambda\bar{\lambda} = \frac{1}{\sqrt{2}}L^{+'\mu}\ \gamma_\mu = \frac{1}{\sqrt{2}}\left(X^{+'}P^\mu - P^{+'}X^\mu\right)\gamma_\mu. \tag{2.20}$$

We should also note the twistor relations that follow from the other three blocks. In three or four dimensions a single doublet λ satisfies this equation automatically. In higher dimensions a single column λ cannot do it automatically, instead we have $\frac{s_d}{4}$ columns in λ with certain relations among them. Thanks to the relations among columns, that can be expressed in terms of gamma matrices as in Eqs.(2.38) below, the equation above will be satisfied.

Next we consider $\left(\bar{\Gamma}\cdot X\right)J$ and use the different gauge fixed forms of the gauge invariant J to show that it vanishes as follows

$$\begin{aligned}\left(\bar{\Gamma}\cdot X\right)J = \left(\bar{\Gamma}\cdot X\right)L &= \frac{1}{4i}\left(\bar{\Gamma}\cdot X\right)\left(\Gamma\cdot X\ \bar{\Gamma}\cdot P - \Gamma\cdot P\ \bar{\Gamma}\cdot X\right) \\ &= \frac{1}{2i}X\cdot X\left(\bar{\Gamma}\cdot P\right) - \frac{1}{4i}X\cdot P\ \left(\bar{\Gamma}\cdot X\right) = 0.\end{aligned} \tag{2.21}$$

The last zero is because $X\cdot P = P\cdot P = 0$ in the Sp$(2,R)$ gauge invariant physical sector. Similarly we show also $\left(\bar{\Gamma}\cdot P\right)J = 0$. Therefore

$$\left(\bar{\Gamma}\cdot P\right)J = 0,\ \left(\bar{\Gamma}\cdot X\right)J = 0. \tag{2.22}$$

Hence, every column of J is a simultaneous null eigenstate of the matrices $\left(\bar{\Gamma}\cdot X\right)$ and $\left(\bar{\Gamma}\cdot P\right)$.

Furthermore, because J can be written in the form $J = Z\bar{Z}$, it must be that the $s_d \times \frac{s_d}{4}$ matrix Z is a collection of these null eigenstates, so it is possible to write Z as a linear combination of $\frac{s_d}{4}$ columns of $L = J$ as follows

$$Z_A^{\ a} = \begin{pmatrix} \mu \\ \lambda \end{pmatrix} = L \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}, \tag{2.23}$$

where α_i are $\frac{s_d}{4} \times \frac{s_d}{4}$ matrices that will be determined below by requiring $J = L = Z\bar{Z}$. So Z must also satisfy the null conditions

$$\left(\bar{\Gamma} \cdot P\right) Z = 0, \quad \left(\bar{\Gamma} \cdot X\right) Z = 0 \text{ and } \ LZ = 0. \tag{2.24}$$

The last null relation $LZ = 0$ is satisfied automatically if the first two are satisfied. This property is sufficient to determine $Z_A^{\ a}$ up to gauge transformations as follows. The explicit matrices are obtained by using the gamma matrices in footnote (6)

$$\left(\bar{\Gamma} \cdot X\right) = \begin{pmatrix} X_\mu \gamma^\mu & -i\sqrt{2}X^{-'} \\ -i\sqrt{2}X^{+'} & -X_\mu \bar{\gamma}^\mu \end{pmatrix}, \quad \left(\bar{\Gamma} \cdot P\right) = \begin{pmatrix} P_\mu \gamma^\mu & -i\sqrt{2}P^{-'} \\ -i\sqrt{2}P^{+'} & -P_\mu \bar{\gamma}^\mu \end{pmatrix} \tag{2.25}$$

Then the zero eigenvalue conditions $\left(\bar{\Gamma} \cdot X\right) Z = 0 = \left(\bar{\Gamma} \cdot P\right) Z$ are solved by

$$\mu = -i\frac{X_\mu \bar{\gamma}^\mu}{\sqrt{2}X^{+'}}\lambda = -i\frac{P_\mu \bar{\gamma}^\mu}{\sqrt{2}P^{+'}}\lambda\,, \quad \lambda = i\frac{X_\mu \gamma^\mu}{\sqrt{2}X^{-'}}\mu = i\frac{P_\mu \gamma^\mu}{\sqrt{2}P^{-'}}\mu. \tag{2.26}$$

To show that these expressions are consistent with each other, note that the second set is obtained by inverting the first set as long as $X^2 = P^2 = X \cdot P = 0$ are satisfied in the physical sector. For example, multiply both sides of the equation $\mu = -i\frac{X_\mu \bar{\gamma}^\mu}{\sqrt{2}X^{+'}}\lambda$ by $i\frac{X_\mu \gamma^\mu}{\sqrt{2}X^{-'}}$, then use $\bar{\gamma}^\mu \gamma^\nu + \bar{\gamma}^\nu \gamma^\mu = 2\eta^{\mu\nu}$ and $X^2 = X^\mu X_\mu - 2X^{+'}X^{-'} = 0$, to obtain $\lambda = i\frac{X_\mu \gamma^\mu}{\sqrt{2}X^{-'}}\mu$. So we can concentrate on the consistency of the first set only. The difference between the two expressions for μ must vanish, this implies that λ should satisfy the Dirac-like equation

$$\left(X^{+'}P^\mu - P^{+'}X^\mu\right)\bar{\gamma}_\mu \lambda = L^{+'\mu}\ \gamma_\mu \lambda = 0. \tag{2.27}$$

This is a consistent equation provided the vector $L^{+'\mu}$ is null

$$L^{+'\mu}L^{+'}_{\ \mu} = \left(X^{+'}P^\mu - P^{+'}X^\mu\right)^2 = 0. \tag{2.28}$$

This is indeed correct in the physical sector that satisfies $X^2 = P^2 = X \cdot P = 0$. Then we can solve for λ by writing

$$\lambda = \left(X^{+'}P^\mu - P^{+'}X^\mu\right)\gamma_\mu \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{2.29}$$

This satisfies the λ equation (2.27) automatically for any $\frac{s_d}{4} \times \frac{s_d}{4}$ matrices α, β. These in turn are determined, up to $\frac{s_d}{4} \times \frac{s_d}{4}$ gauge transformations, by satisfying the $\frac{s_d}{4} \times \frac{s_d}{4}$ matrix equation (2.20).

In this way we have seen that all of the forms given in Eq.(2.26) are consistent with each other and determine $Z_A^{\ a}$ up to a gauge transformation. Thus the null conditions in Eq.(2.24) is all that is needed to determine $Z_A^{\ a}$ up to a gauge transformation, but in turn these followed from the basic relation in Eq.(2.19).

The following subset of our relations resemble the twistor transform in four dimensions, but μ, λ are $\frac{s_d}{2} \times \frac{s_d}{4}$ matrices that must obey all the relations above

$$\mu = -i\frac{X_\mu \bar{\gamma}^\mu}{\sqrt{2}X^{+'}}\lambda, \text{ and } \lambda\bar{\lambda} = \frac{1}{\sqrt{2}}\left(X^{+'}P^\mu - P^{+'}X^\mu\right)\gamma_\mu. \tag{2.30}$$

Indeed we can check directly that by inserting only these relations into the right hand side of Eqs.(2.19,2.17), we derive the SO$(d,2)$ generators in terms of phase space $L^{MN} = X^M P^N - X^N P^N$ that appear on the left side of those equations. Furthermore by inserting only these relations into the twistor action we derive the phase space action that determines the canonical structure

$$\frac{4}{s_d}\int d\tau\ i\partial_\tau Z_A^{\ a}\bar{Z}_a^{\ A} = i\frac{4}{s_d}\int d\tau\ Tr\left(\partial_\tau\mu\bar{\lambda} + \partial_\tau\lambda\bar{\mu}\right) = \frac{4}{s_d}\int d\tau\ Tr\left(\partial_\tau\left(\frac{X_\mu\bar{\gamma}^\mu}{\sqrt{2}X^{+'}}\right)\lambda\bar{\lambda}\right) \tag{2.31}$$

$$= \frac{4}{s_d}\int d\tau\ \frac{1}{\sqrt{2}}\left(X^{+'}P^\mu - P^{+'}X^\mu\right)Tr\left(\partial_\tau\left(\frac{X_\mu\bar{\gamma}^\mu}{\sqrt{2}X^{+'}}\right)\gamma_\mu\right) \tag{2.32}$$

$$= \int d\tau\ \left(X^{+'}P_\mu - P^{+'}X_\mu\right)\partial_\tau\left(\frac{X^\mu}{X^{+'}}\right) \tag{2.33}$$

$$= \int d\tau\ \left(P_\mu - \frac{P^{+'}}{X^{+'}}X_\mu\right)\left(\partial_\tau X^\mu - \frac{\partial_\tau X^{+'}}{X^{+'}}X^\mu\right) \tag{2.34}$$

$$= \int d\tau\ \left(\partial_\tau X^\mu P_\mu - \partial_\tau X^{+'}P^{-'} - \partial_\tau X^{-'}P^{+'}\right) = \int d\tau\ \partial_\tau X^M P_M. \tag{2.35}$$

The last line follows when the constraints $X^2 = P^2 = X\cdot P = 0$ are satisfied in the physical sector. This shows the consistency of our twistor transform of Eq.(2.30) for spinless particles in all dimensions.

It is also interesting to give an explicit formula for both μ and λ in terms of the Sp$(2,R)$ gauge invariant L^{MN}. This is already obtained through the relation between Z and L given in Eq.(2.23). It turns out that it is sufficient to take only one of the α_i to be nonzero. So we will take $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and determine $\alpha_1 \neq 0$ from the relation $J = L = Z\bar{Z}$. The other possibilities are gauge equivalent. The equivalence is guaranteed by the Sp$(2,R)$ gauge invariance conditions $X^2 = P^2 = X\cdot P = 0$. This gives

$$Z_A^{\ a} = \begin{pmatrix}\mu\\ \lambda\end{pmatrix} = L\begin{pmatrix}\alpha_1\\ 0\\ 0\\ 0\end{pmatrix} \tag{2.36}$$

We write out the $L=J$ in Eq.(2.19) more explicitly in terms of $\frac{s_d}{4}\times\frac{s_d}{4}$ blocks by using the gamma matrices in footnote (6), and we obtain

$$\mu=\begin{pmatrix} L^{+'-'}+L^{r0}\rho_r+iL^{12}+\frac{1}{2}L_{rs}\rho^{rs} \\ -L^{01}-iL^{02}-\left(L^{r1}+iL^{r2}\right)\rho_r \end{pmatrix} hu, \tag{2.37}$$

$$\lambda=\begin{pmatrix} L^{+'0}+L^{+'r}\ \rho_r \\ L^{+'1}+iL^{+'2} \end{pmatrix} i\sqrt{2}hu. \tag{2.38}$$

Here $\alpha_1=hu$ is written as a product of a unitary matrix u and a Hermitian matrix h. The matrix u is an arbitrary $\frac{s_d}{4}\times\frac{s_d}{4}$ unitary matrix that belongs to the gauge group that acts on the twistors. The generator of this gauge group is the constraint $\left(\bar{Z}Z\right)_a^{\ b}=0$. The matrix h is determined by insuring $\lambda\bar{\lambda}=\frac{1}{\sqrt{2}}L^{+'\mu}\gamma_\mu$ that was established in Eq.(2.20) and is given by

$$h=2^{-3/4}\left(L^{+'0}+L^{+'r}\ \rho_r\right)^{-1/2}. \tag{2.39}$$

To summarize, in Eqs.(2.37-2.39) we have given a more fundamental form of the twistor transform between $Z_A^{\ a}$ and the $(d+2)$ dimensional phase space $\left(X^M,P^M\right)$. The formulas are Sp$(2,R)$ gauge invariant since (X,P) appear only in the form $L^{MN}=X^MP^N-X^NP^M$. These transforms are consistent with Eqs.(2.30,2.26,2.24) which somewhat resemble the more traditional form of the twistor transform.

Note that $Z\left(L\right)$ written in terms of L^{MN} depends only on $2\left(d-1\right)$ independent combinations of the L^{MN}, since the L^{MN} obey the constraints $L^{MN}L_{NK}=0$ that follow from $X^2=P^2=X\cdot P=0$ in the physical sector. From these we derive the explicit relations

$$\left(L^{+'-'}\right)^2=-L^{+'\mu}L^{-'}_{\ \ \mu},\ L^{\mu\nu}=\frac{L^{+'\mu}L^{-'\nu}-L^{+'\nu}L^{-'\mu}}{L^{+'-'}},\ L^{+'\mu}L^{+'}_{\ \ \mu}=0=L^{-'\mu}L^{-'}_{\ \ \mu}.$$

So, all L^{MN} are then written only in terms of the $2d$ vector components $L^{\pm'\mu}$, but those are lightlike vectors and therefore contain only $2\left(d-1\right)$ independent degrees of freedom.

Both the 2T particle and the corresponding twistors are SO$(d,2)$ covariant descriptions, the first is in terms of vectors $X_i^M=\left(X^M,P^M\right)$ and the second is in terms of spinors $Z_A^{\ a}$ of SO$(d,2)$. This covariance is achieved by having gauge symmetries in both versions, in the first case the gauge symmetry is applied on the i index of X_i^M and in the second case the gauge symmetry is applied on the a index of $Z_A^{\ a}$.

To account for the gauge symmetry and the constraint (2.16) we may derive them from a twistor action principle. This is done by rewriting the twistor action in Eq.(2.14) in a form similar to Eq.(1.7)

$$S_{twistor}=\frac{4}{s_d}\int d\tau\ Tr\left[\left(i\bar{Z}DZ\right)-h_dV\right],\ \ \left(DZ_A\right)^a\equiv\frac{\partial Z_A^{\ a}}{\partial\tau}-iZ_A^{\ b}V_b^{\ a}. \tag{2.40}$$

The equation of motion of the $\frac{s_d}{4}\times\frac{s_d}{4}$ matrix gauge field $V_b{}^a$ generates the constraint $\left(\bar{Z}Z\right)_a{}^b - h_d\delta_a{}^b = 0$ where h_d is chosen to make the matrix traceless or with trace depending on the number of dimensions d (see e.g. the counting of degrees of freedom for $d = 5, 6$ following Eq.(2.16)). Equivalently, the matrix V itself can be taken as traceless or with trace depending on the dimension, and this will result in the same constraint. Once this constraint is satisfied this action reduces to the previous one in Eq.(2.14). For $d = 3, 4, 5, 6$ this is the full action principle in terms of twistors since there are no other conditions (other than reality or pseudo-reality in some dimensions) as discussed following Eq.(2.16). However, for $d \geq 7$ some more conditions on a general complex or (pseudo)real Z are required to make it satisfy also the basic relation Eq.(2.17) or its equivalent null conditions in Eq.(2.24). Although we have given the full twistor transform for any dimension, we have so far given the full action principle only for $d \leq 6$.

Up to now we have described the twistor transform for the "free" 2T particle in $d+2$ dimensions. But from here it is an easy step to obtain the twistor transform for an assortment of non-trivial particle dynamics in 1T-physics. Having established the transform between twistors and the $\mathrm{Sp}(2,R)$ doublets $\left(X^M, P^M\right)$ in $(d+2)$ dimensions, we can now make $\mathrm{Sp}(2,R)$ gauge choices to produce various dynamical systems in $(d-1)+1$ dimensions, in the physical sector that satisfies $X^2 = P^2 = X\cdot P = 0$. Examples that occur in this paper are the massless particle in d dimensions of Eq.(1.22,1.23), or the massive particle in d dimensions of Eq.(1.25,1.26). Other $\mathrm{Sp}(2,R)$ gauge choices that include interacting and curved background cases are found in [4][5]. By inserting the gauge choice for $\left(X^M, P^M\right)$ into Eqs.(2.30,2.26) we obtain the corresponding twistors in d dimensions, such as the twistors for the massless particle of Eq.(1.8), or the massive particle in of Eq.(1.14). For more examples see [7] where the computations for the twistor transform were done explicitly in $d = 4$, but the same explicit formulas also apply in d dimensions by inserting the corresponding gamma matrices in d dimensions, as given in the expressions above.

C. Geometry: twistors as the coset $\mathbf{SO}(d,2)/\mathbf{H}_\Gamma$

A geometric view of twistors in d dimensions can also be given in the form of a coset as follows. The starting point for twistors was the twistor gauge of 2T-physics in Eq.(2.14) which involved the group element $g(\tau)$ in the spinor representation of $\mathrm{SO}(d,2)$ and the special matrix Γ in Eq.(2.13). The action and its $\mathrm{SO}(d,2)$ global symmetry charge (on the left side of g) have the form

$$S(g) = \frac{4}{s_d}\int d\tau\ Tr\left(ig^{-1}\partial_\tau g\Gamma\right),\quad (J)_A^B = \left(g\Gamma g^{-1}\right)_A^B. \tag{2.41}$$

The action is like a sigma model, but it is linear instead of being quadratic in the Cartan connection $ig^{-1}\partial_\tau g$, and has the special insertion Γ on the right side of g. The insertion Γ determines important properties of this action. The equation of motion for g is $[\Gamma, g^{-1}\partial_\tau g] = 0$. Using this one can show that the global current is conserved $\partial_\tau \left(g\Gamma g^{-1}\right) = 0$, as expected from Noether's theorem.

We recall that the current $J_A{}^B$ is gauge invariant and contains all the physical information of the theory as seen in the previous sections. In particular the current satisfies $J^2 = \left(g\Gamma g^{-1}\right)\left(g\Gamma g^{-1}\right) = \left(g\Gamma^2 g^{-1}\right) = 0$ at the classical level, which is consistent with the covariant approach in section (II A). This property of the current captures all the essential aspects of the physical sector at the classical level.

Now, let us determine the independent degrees of freedom that contribute to the current. We will find that there are precisely $2(d-1)$ degrees of freedom, precisely equal to the number of physical degrees of freedom. Since $g^{-1}\Gamma g$ is a $\mathrm{SO}(d,2)$ transformation applied on a generator Γ in the algebra of $\mathrm{SO}(d,2)$, we can eliminate from $g(\tau)$ the subgroup H_Γ that leaves Γ invariant, and keep only the coset degrees of freedom in $\mathrm{SO}(d,2)/\mathrm{H}_\Gamma$. To do this, we can decompose $g(\tau) = T_\Gamma(\tau) H_\Gamma(\tau)$ and write $g\Gamma g^{-1} = T_\Gamma \Gamma T_\Gamma^{-1}$ since by definition $H_\Gamma \Gamma H_\Gamma^{-1} = \Gamma$. Here $H_\Gamma = \exp(h_\Gamma)$ and $T_\Gamma = \exp(t_\Gamma)$, where h_Γ (t_Γ) is a linear combination of all the $\mathrm{SO}(d,2)$ generators Γ^{MN} that commute (do not commute) with Γ, i.e. $[h_\Gamma, \Gamma] = 0$ and $[t_\Gamma, \Gamma] \neq 0$.

To characterize the sets of generators (h_Γ, t_Γ) consider the decomposition of $\mathrm{SO}(d,2)$ with respect to the $\mathrm{SO}(d-2)\times\mathrm{SO}(2,2)$ subgroup. We can write $J^{MN} = J^{ij} \oplus J^{\mu\nu} \oplus J^{\mu i}$ where $i = 1, 2, \cdots, (d-2)$ spans the $\mathrm{SO}(d-2)$ basis and $\mu = +', -', +, -$ (or $0', 0, 1', 1$) spans the $\mathrm{SO}(2,2)$ basis. Furthermore we decompose $\mathrm{SO}(2,2)$ $=\mathrm{SL}(2,R)_+ \times \mathrm{SL}(2,R)_-$ and note that each μ index is in the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation of $\mathrm{SL}(2,R)_+ \times \mathrm{SL}(2,R)_-$. The $\mathrm{SL}(2,R)_+ \times \mathrm{SL}(2,R)_-$ generators can be identified explicitly as

$$\mathrm{SL}(2,R)_+ : \ \frac{1}{2}\left(J^{+'-'} + J^{+-}\right), \ J^{+'+}, \ J^{-'-} \tag{2.42}$$

$$\mathrm{SL}(2,R)_- : \ \frac{1}{2}\left(J^{+'-'} - J^{+-}\right), \ J^{+'-}, \ J^{-'+} \tag{2.43}$$

From the general $\mathrm{SO}(d,2)$ commutation rules

$$\left[J^{MN}, J^{KL}\right] = i\left[\left(J^{ML}\eta^{NK} - (M \leftrightarrow N)\right) - (K \leftrightarrow L)\right], \tag{2.44}$$

with $\eta^{+'-'} = \eta^{+-} = -1$ and $\eta^{ij} = \delta^{ij}$, it is easy to verify that these indeed form the $\mathrm{SL}(2,R)_+ \times \mathrm{SL}(2,R)_-$ algebra. Under commutation with the generator $\frac{1}{2}\left(J^{+'-'} + J^{+-}\right)$ each $\mathrm{SO}(d,2)$ generator has a definite charge $0, \pm\frac{1}{2}, \pm 1$. We list the generators according to those

charges as follows

$$\begin{array}{ccccc} -1 & -\frac{1}{2} & 0 & +\frac{1}{2} & +1 \\ J^{-'-} & \begin{pmatrix} J^{-i} \\ J^{-'i} \end{pmatrix} & \begin{array}{c} J^{ij},\ \frac{J^{+'-'}+J^{+-}}{2} \\ \left(J^{-'+},\frac{J^{+'-'}-J^{+-}}{2},J^{+'-}\right) \end{array} & \begin{pmatrix} J^{+'i} \\ J^{+i} \end{pmatrix} & J^{+'+} \end{array} \tag{2.45}$$

These charges are conserved additively in the general commutation rules $\left[J^{MN},J^{KL}\right]=\cdots$ given above. Furthermore the charge $\pm\frac{1}{2}$ generators form doublets under $\mathrm{SL}(2,R)_-$ and vectors under $\mathrm{SO}(d-2)$ as indicated, while the charge ± 1 generators are singlets under both. From this structure of the commutation rules $\left[J^{MN},J^{KL}\right]=\cdots$ we easily see that the generators that commute or do not commute with $\Gamma\sim\Gamma^{-'-}$ (or $J^{-'-}$) are

$$h_\Gamma:\ J^{-'-},\ \begin{pmatrix} J^{-i} \\ J^{-'i} \end{pmatrix},\ \begin{array}{c} J^{ij} \\ \left(J^{-'+},\frac{J^{+'-'}-J^{+-}}{2},J^{+'-}\right) \end{array} \tag{2.46}$$

$$t_\Gamma:\ \frac{1}{2}\left(J^{+'-'}+J^{+-}\right),\ \begin{pmatrix} J^{+'i} \\ J^{+i} \end{pmatrix},\ J^{+'+} \tag{2.47}$$

We note that each set forms a subalgebra $[h_\Gamma,h_\Gamma]\sim h_\Gamma,\ [t_\Gamma,t_\Gamma]\sim t_\Gamma$ while $[h_\Gamma,t_\Gamma]\sim h_\Gamma+t_\Gamma$. In particular, within t_Γ the generator $\frac{1}{2}\left(J^{+'-'}+J^{+-}\right)$ forms a $\mathrm{U}(1)$ subgroup and classifies the others according to their charges above, while the remaining coset $T_\Gamma/U(1)$ forms an algebra similar to the Heisenberg algebra, $\left[J^{+'i},J^{+j}\right]=i\delta^{ij}J^{+'+}$, since $J^{+'+}$ commutes with both $J^{+'i},J^{+j}$. From this we conclude that the general $T_\Gamma=\exp(t_\Gamma)$ can be parametrized as follows

$$T_\Gamma=\exp(t_\Gamma)=\exp\left(\frac{1}{2}\left(\Gamma^{+'-'}+\Gamma^{+-}\right)\omega(\tau)\right)\exp\left(\Gamma^{+'+}z(\tau)\right)\exp\left(\Gamma^{+i}k_i(\tau)+\Gamma^{+'i}q_i(\tau)\right) \tag{2.48}$$

The number of parameters in this coset is precisely $2(d-1)$, which is the same as the number of physical degrees of freedom. If we insert the explicit set of gamma matrices Γ^{MN} given in footnote (6) into this expression, we can write T_Γ in the form of $\frac{s_d}{4}\times\frac{s_d}{4}$ similar to Γ. Then, as seen from Eq.(2.14), the first block of columns that forms an $s_d\times\frac{s_d}{4}$ rectangular matrix is the twistor $Z_A^{\ a}(t_\Gamma)$ now written in terms of only the $2(d-1)$ parameters of the coset $t_\Gamma\in\mathrm{SO}(d,2)/\mathrm{H}_\Gamma$.

This result is of course in agreement with the form of the $\mathrm{Sp}(2,R)$ invariant $Z_A^{\ a}(L)$ of Eqs.(2.37,2.38), which can also be written only in terms of $2(d-1)$ parameters as detailed in Sec.(**??**). We now understand that there is a close relationship with the geometric interpretation as a coset.

III. D-BRANES AND TWISTORS

In the discussion following the twistor action in Eq.(2.40) we explained that more conditions are needed on Z in order to obtain the proper twistor that is equivalent to the phase space of a particle when $d \geq 7$. What if those conditions are never imposed? What would then be the content of $Z_A^{\ a}$? We find that the extra degrees of freedom can be interpreted as collective coordinates of D-branes.

To see this let us consider the properties of the rectangular matrix $Z_A^{\ a}$ that follow from the action in Eq.(2.40). The gauge group acts on the right side and there is a global symmetry with conserved charges J that acts on the left side. These properties are summarized by the equations

$$\bar{Z}Z = 0,\ J = Z\bar{Z} - trace. \tag{3.1}$$

Thus, the global current J is a $s_d \times s_d$ matrix in the fundamental representation of the global group G that can be expanded in a complete set of SO$(d,2)$ gamma matrices as follows

$$J = Z\bar{Z} = J_0 + \Gamma^M J_M + \frac{1}{2}\Gamma^{MN} J_{MN} + \frac{1}{3!}\Gamma^{MNK} J_{MNK} + \cdots \tag{3.2}$$

There are as many terms $\Gamma^{M_1 \cdots M_n}$ as necessary to span all the generators of some group G whose fundamental representation has the same dimension s_d of the SO$(d,2)$ spinor. If the group is constrained to be SO$(d,2)$ there is only the term proportional to Γ^{MN} in the expansion (3.2), and then the group element g is constructed by exponentiating the generators as in Eq.(2.1). But if the group is more general, then the exponent in Eq.(2.1) contains all the terms that appear in Eq.(3.2).

Thus, with a more general g represented as a $s_d \times s_d$ matrix, a more general twistor Z would emerge, with more degrees of freedom than the particle phase space. This is achieved with an action that is written in the form of Eq.(2.41), which in turn is obtained by gauge fixing from the 2T-physics action in (2.2), but by taking g to be a group element not just in SO$(d,2)$, but an element in the smallest group G that contains SO$(d,2)$ in the spinor representation. The parent 2T-physics theory in Eq.(2.2) has an interaction term of the form of $\frac{4}{s_d} Tr\,(ig^{-1}\partial_\tau g L)$ which is unchanged. But we emphasize that now L is proportional to only Γ^{MN}, while $ig^{-1}\partial_\tau g$ has all the terms in Eq.(3.2), so L couples to only the SO$(d,2)$ subgroup of G. Then the action in Eq.(2.2) still has local Sp$(2,R)$ and SO$(d,2)$ symmetries, but now it has global G symmetry instead of only global SO$(d,2)$ on the left side of g.

This generalization of the group element g allows Z to contain the extra degrees of freedom. We emphasize that the spinor representation of SO$(d,2)$, whose dimension is s_d, must correspond to the fundamental representation of G. This requirement determines G as we see in Table 1 below.

We already know that for $d = 3, 4, 6$ the groups G =Sp$(4, R)$, SU$(2, 2)$, Spin$(6, 2)$ respectively are exactly equal to SO$(d, 2)$ in the spinor representation. Therefore for these cases there are no other terms in Eq.(3.2) other than Γ^{MN}, provided some (pseudo)reality conditions are imposed as given following Table 1 below. Extra terms usually appear for $d \geq 7$.

As an example of what terms appear, consider SO$(7, 2)$ for $d = 7$. The spinor representation has dimension 16. The smallest group with a 16 dimensional fundamental representation is SO* (16), where the * indicates the appropriate analytic continuation that contains SO$(7, 2)$ as a subgroup. The number of generators of SO* (16) is $\frac{16 \cdot 15}{2} = 120$. The number of generators represented by the 16×16 gamma matrices is $\Gamma^M \rightarrow 9$, $\Gamma^{MN} \rightarrow \frac{9 \cdot 8}{2} = 36$, $\Gamma^{MNK} \rightarrow \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$, $\Gamma^{MNKL} \rightarrow \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126$. We see that the 120 generators of SO* (16) are represented by $\frac{1}{2}\Gamma^{MN} L_{MN} + \frac{1}{3!}\Gamma^{MNK} L_{MNK}$. Therefore the more general twistor in $d = 7$ has the expansion

$$d = 7: \; J = Z\bar{Z} = \frac{1}{2} L^{MN} \Gamma_{MN} + \frac{1}{3!} L^{MNK} \Gamma_{MNK}. \tag{3.3}$$

If we impose additional conditions on Z as in the previous sections, then we eliminate the term $L_{MNK} = 0$, and the remaining L_{MN} necessarily satisfies all the conditions of section (II B) since they all followed from $\bar{Z}Z = 0$ that was imposed by the gauge symmetry H.

However, if we do not impose the conditions of section (II B) then we can interpret the degrees of freedom L_{MNK} as D-brane degrees of freedom. To see this, consider the smallest extended super-*conformal* algebra that contains spin$(7, 2)$ $\subset$SO* (16). This is OSp$(16|2)$. Its two supercharges satisfy $\{Q_A^i, Q_B^j\} = \varepsilon^{ij} \left[\frac{1}{2}\left(\Gamma^{MN}\right)_{AB} L_{MN} + \frac{1}{3!}\left(\Gamma^{MNK}\right)_{AB} L_{MNK}\right] + q^{ij} C_{AB}$, with $i = 1, 2$ labelling the Sp(2), and C_{AB}, q^{ij} both symmetric. The usual $d = 7$ Poincaré super-algebra is a subalgebra obtained from the above by decomposing the SO(7,2)→SO(6,1)×SO(1,1) for spinors ($16 = 8_+ + 8_-$) as $A = \alpha_+ \oplus \alpha_-$ and vectors as $M = \pm', \mu$, and keeping only the operators $Q_{\alpha+}^i$ and all the L^{MN}, L^{MNK} with a single $+'$ as follows

$$\left\{Q_{\alpha+}^1, Q_{\beta+}^2\right\} = L^{+'\mu} \left(\Gamma_{+'\mu}\right)_{\alpha\beta} + \frac{1}{2} L^{+'\mu\nu} \left(\Gamma_{+'\mu\nu}\right)_{\alpha\beta}. \tag{3.4}$$

In the massless particle gauge, $L^{+'\mu}$ is the momentum p^μ and then $L^{+'\mu\nu}$ are the *commuting* D2-brane charges in $d = 7$ dimensions (like generalized momenta). The other components of L_{MN}, and L_{MNK} are functions of phase space, including the particle as well as D-brane canonical degrees of freedom, and do not generally commute among themselves[8].

As another example consider SO$(8, 2)$ for $d = 8$. The two spinor representations are $16, \overline{16}$. The smallest groups with 16 dimensional fundamental representations are SO* (16), Sp* (16),

[8] The commutation rules of the $L^{M_1 \cdots M_n}$ are isomorphic to the commutation rules of the $\Gamma^{M_1 \cdots M_n}$. From this we see that the D-brane charges $L^{+'\mu\nu}$ commute among themselves as well as with the momenta $p^\mu = L^{+'\mu}$.

$SU^*(16)$. To decide which is the smallest one that contains $SO(8,2)$ as a subgroup we analyze the number of generators represented by the gamma matrices. The number of generators of $SO^*(16)$ is $\frac{16\cdot 15}{2}=120$, for $Sp^*(16)$ is $\frac{16\cdot 17}{2}=136$, and for $SU^*(16)$ is $(16)^2-1=255$. The number of generators represented by the gamma matrices is $\Gamma^M\to 10$, $\Gamma^{MN}\to\frac{10\cdot 9}{2}=45$, $\Gamma^{MNK}\to\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}=120$, $\Gamma^{MNKL}\to\frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}=210$, $\Gamma_+^{MNKLR}\to\frac{1}{2}\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5}=126$, where the last one is self dual (hence the extra factor of $\frac{1}{2}$). The 45 generators of $SO(8,2)$ represented by Γ^{MN} must be included as one of the criteria in choosing the smallest G. Then we see that there is no combination of gamma matrices that can be used to construct $SO^*(16)$ and $Sp^*(16)$ and therefore we must take $SU^*(16)$ as the smallest group that contains $spin(8,2)$. The smallest superconformal algebra is $SU(16|1)$. The 255 generators of $SU^*(16)$ that appear in $\left\{Q_A,\bar{Q}^B\right\}$ are then represented by $\frac{1}{2}\Gamma^{MN}L_{MN}+\frac{1}{4!}\Gamma^{MNKL}L_{MNKL}$. The extra L_{MNKL} lead to the D-brane degrees of freedom. To see the content of D-brane *commuting*[8] charges we must decompose $SO(8,2)$ to $SO(7,1)$ by $M=\pm',\mu$ and identify the D-brane commuting charges as the D3-brane $Z^{+'\mu\nu\lambda}$ in 8 dimensions.

d	Spin$(d,2)$	spinor	G	G$_{\text{super}}$ (N)	generators of G in Spin$(d,2)$basis	contained in product
3	$Spin(3,2)$	4	$Sp(4,R)$	$OSp(N\|4)$	$\Gamma^{MN}\ _{10}$	$(4\times 4)_s$
4	$Spin(4,2)$	$4,\bar{4}$	$SU(2,2)$	$SU(2,2\|N)$	$\Gamma^{MN}\ _{15}$	$4\times\bar{4}$
5	$Spin(5,2)$	8_+	$spin^*(7)$ $SO^*(8)$	$F(4)$ $OSp(8\|2N)$	$\Gamma^{MN}\ _{21}$ $\Gamma^{MN}\ _{21}\oplus\ \Gamma^{M}\ _{7}$	$(8\times 8)_a$
6	$Spin(6,2)$	8_+	$SO^*(8)$	$OSp(8\|2N)$	$\Gamma^{MN}\ _{28}$	$(8\times 8)_a$
7	$Spin(7,2)$	16	$SO^*(16)$	$OSp(16\|2N)$	$\Gamma^{MN}\ _{36}\oplus\Gamma^{MNK}\ _{84}$	$(16\times 16)_a$
8	$Spin(8,2)$	$16,\overline{16}$	$SU^*(16)$	$SU(16\|N)$	$\Gamma^{MN}\ _{45}\oplus\Gamma^{MNKL}\ _{210}$	$16\times\ \overline{16}$
9	$Spin(9,2)$	32	$Sp^*(32)$	$OSp(N\|32)$	$\Gamma^{MN}\ _{55}\oplus\Gamma^{M}\ _{11}\oplus\Gamma^{M_1\cdots M_5}\ _{462}$	$(32\times 32)_s$
10	$Spin(10,2)$	32_+	$Sp^*(32)$	$OSp(N\|32)$	$\Gamma^{MN}\ _{66}\oplus\Gamma_+^{M_1\cdots M_6}\ _{462}$	$(32\times 32)_s$
11	$Spin(11,2)$	64	$Sp^*(64)$	$OSp(N\|64)$	$\Gamma^{MN}\ _{78}\oplus\Gamma^{MNK}\ _{286}\oplus\Gamma^{M_1\cdots M_6}\ _{1716}$	$(64\times 64)_s$
12	$Spin(12,2)$	$64,\overline{64}$	$SU^*(64)$	$SU(64\|N)$	$\Gamma^{MN}\ _{91}\oplus\Gamma^{MNKL}\ _{1001}\oplus\Gamma^{M_1\cdots M_6}\ _{3003}$	$64\times\ \overline{64}$

Table 1: Smallest groups G and supergroups G_{super} that contain Spin$(d,2)$, and D-branes.

In Table-1 we give a list of the smallest groups G that contain $spin(d,2)$ for $3\leq d\leq 12$. We also include the smallest supergroup G_{super} that contains G. We list the gamma matrix representation of the generators of G, and their numbers as subscripts, as represented by antisymmetrized products of gamma matrices $\Gamma^{M_1\cdots M_n}\equiv\frac{1}{n!}\left(\Gamma^{M_1}\bar{\Gamma}^{M_2}\Gamma^{M_3}\cdots\Gamma^{M_n}\mp\text{permutations}\right)$ in dimension $d+2$ labelled by M. The last column gives information on whether the gamma matrices occur in the symmetric or antisymmetric products of the spinors of $SO(d,2)$, when both spinor indices A,B are lowered or raised in the form $\left(\Gamma^{M_1\cdots M_n}\right)_{AB}$ by using the metric C in spinor space. The phase space of the D-branes correspond to the extra generators beyond Γ^{MN} as explained in the examples above. In the case of $d=5$, one option is to keep the D0-brane associated with $\Gamma^M\to\Gamma^{+'}$, another option is to remove it with the extra

U(1) gauge symmetry as discussed in the counting done in section (II B). The D-brane does not occur for the supergroup F(4) that can be used for the $d=5$ superparticle as described below.

Groups that are larger than the listed G may be considered in our scheme in every dimension (e.g. SU(8) instead of SO(8) in $d=6$, etc.). In that case the number of generators $\Gamma^{M_1\cdots M_n}$ increases compared to the ones listed in the table for each d. Furthermore the corresponding D-brane degrees of freedom also get included in the model.

When $Z_A^{\ a}$ is obtained from the group element g through the relation $Z\bar{Z}=g\Gamma g^{-1}$, with the group G listed in the table above, then Z is real or pseudo-real when the group is SO or Sp and it is complex when the group is SU. Given those properties, in general the quadratic $\left(Z\bar{Z}\right)_A^{\ B}$ contains just the gamma matrices listed above which correspond to the generators of the group G. If the (pseudo)reality properties associated with G are not obeyed by Z then more D-brane terms will appear generally in the expansion of $Z\bar{Z}$ as in Eq.(3.2) as compared to those on the table.

The case of $d=11$ is particularly interesting since it relates to M-theory as follows. The corresponding twistors are spinors of Spin$(11,2)$ that are 64 dimensional. The smallest group is Sp* (64) whose generators are represented by Γ^{MN} (78) + Γ^{MNK} (286) + $\Gamma^{M_1\cdots M_6}$ (1716). To identify the commuting charges of D-branes we decompose $M=\pm',\mu$ and keep all the generators with a single $+'$, as follows $L^{+'\mu}\oplus L^{+'\mu\nu}\oplus L^{+'\mu_1\cdots\mu_5}$. Here $L^{+'\mu}$ is the momentum in 11 dimensions and $L^{+'\mu\nu},L^{+'\mu_1\cdots\mu_5}$ are the D2-brane and D5-brane *commuting*[8] charges respectively.

Let us mention that the discussion above with the group G can be directly generalized to the supergroup $G_{\text{super}}\left(N\right)$ listed in Table 1 (with some limits on N as discussed below) by following [6]. The 2T-physics action is still of the same form as $S\left(X,P,g\right)$ of Eq.(2.2), but now we have a supergroup element g and a supertrace coupling $\frac{4}{s_d}Tr\left(ig^{-1}\partial_\tau gL\right)$, and the matrix L is of the form $L=\frac{1}{4i}\begin{pmatrix}\Gamma^{MN}&0\\0&0\end{pmatrix}L_{MN}$ where the gamma matrices couple to the bosonic subgroup G as above. This 2T superparticle action reduces to the standard massless superparticle action in the particle gauge for dimensions $d=3,4,5,6$ [6] with N supersymmetries. It can also be gauge fixed to the twistor gauge to give supertwistors that are equivalent to the super phase space in those dimensions [12][13]. One can go beyond those gauge choices and obtain a twistor description of many other dual superparticle theories that give the super generalizations of the ones studied recently in [7].

The supergroup can be enlarged to have more fermionic generators, but keeping G as a bosonic subgroup. For example, for $d=4$ we may take SU$(2,2|N)$ instead of the smallest $N=1$ shown in the table. For physical purposes the total number of real fermionic generators cannot exceed 64 (32 ordinary supercharges and 32 conformal supercharges). For example, for $d=4$ we can go as far as $N=8$, or G_{super} =SU$(2,2|8)$ which has 64 real fermionic

parameters. Similarly, for $d=10$ we may take OSp$(1|32)$ or OSp$(2|32)$. In the more general cases the coupling L can be of the form of the previous paragraph $L=\frac{1}{4i}\begin{pmatrix}\Gamma^{MN} & 0\\ 0 & 0\end{pmatrix}L_{MN}$.

The model can also be generalized by adding d' more dimensions $\left(X^I,P^I\right)$ in addition to the $d+2$ dimensions $\left(X^M,P^M\right)$, but keeping the same $g\in G_{\text{super}}$. The generalized action has the form [18][12][13][17]

$$S_{2T}\left(\hat{X},\hat{P},g\right)=\int d\tau\left[\frac{1}{2}\varepsilon^{ij}\partial_\tau\hat{X}_i\cdot\hat{X}_j-\frac{1}{2}A^{ij}\hat{X}_i\cdot\hat{X}_j+\frac{4}{s_d}Str\left(ig^{-1}\partial_\tau g\hat{L}\right)\right] \tag{3.5}$$

where $\hat{X}^{\hat{M}}=\left(X^M,X^I\right)$, $\hat{P}^{\hat{M}}=\left(P^M,P^I\right)$, and we now take the more general coupling $\hat{L}=\frac{1}{4i}\begin{pmatrix}\Gamma^{MN} & 0\\ 0 & 0\end{pmatrix}L_{MN}+\frac{\alpha}{4i}\begin{pmatrix}0 & 0\\ 0 & \Gamma^{IJ}\end{pmatrix}L_{IJ}$. The fixed parameter $\alpha=\frac{s_d}{s_{d'}}$ is determined by local bosonic and fermionic symmetries in this action, to be the ratio of the spinor dimensions of SO$(d+2)$ and SO(d') . In this latter scheme we obtain interesting cases, such as supertwistors with some compactified subspaces, without D-branes. For example supertwistors for $\mathrm{AdS}_4\times\mathrm{S}^7$, $\mathrm{AdS}_5\times\mathrm{S}^5$, $\mathrm{AdS}_7\times\mathrm{S}^4$ with a of total 10 or 11 dimensions emerge [12][13][17] by using supergroups with only 32 real fermions, namely $G_{\text{super}}=$OSp$(8|4^*)$, SU$(2,2|4)$, OSp$(8^*|4)$ respectively.

This analysis taken to the maximum allowed number of supersymmetries and the maximum number of dimensions leads to $d=11$ with OSp$(1|64)$ as the hidden global supersymmetry for M-theory, and suggests that the extended supertwistors may well play a role in a nice description of M-theory. This can be studied through the toy M-model [19] that has the action $S\left(X,P,g\right)$ of Eq.(2.2) with the group OSp$(1|64)$, and includes the D2 and D5-branes. The supergroup OSp$(1|64)$ is motivated by other considerations as well [20][21][22]. In other approaches to twistors in D=11 [23][24], twistors in the fundamental representation of OSp$(1|64)$ were used in [24] for the formulation of a superstring action in an extended D=11 superspace.

The twistor action (2.40) is a gauge fixed form of the 2T-physics action $S\left(X,P,g\right)$ of Eq.(2.2) for a general g. We can play the game of gauge fixing the local symmetries Sp$(2,R)$ and SO$(d,2)$ of $S\left(X,P,g\right)$ in many possible ways and derive a multitude of 1T-physics systems with a rich web of dualities among them. D-branes are included in this web of dualities. Then twistors can be shown to unify many dual theories including D-branes. It would be interesting to pursue this line of reasoning in more detail.

IV. DISCUSSION

In this paper we gave the general twistor transform that maps twistor space to phase space in d dimensions with one time. The general transform can be specialized to a variety of special dynamical particle systems that include particles with or without mass, relativistic

or nonrelativistic, in flat or curved spaces, interacting or non-interacting. Thus, the scope of our formulas is much larger than the traditional twistor transform. The special cases of phase space described by the same twistor are those that can be derived by gauge fixing the parent unifying theory in 2T-physics. Thus, either the twistor description or the vector $\mathrm{SO}(d,2)$ description in 2T-physics provide a unification of those 1T-physics systems and establishes a duality relationship between them.

To our knowledge this is the first time that twistors have been successfully defined generally in d dimensions. We have insured that our twistor transform is fully equivalent to particle phase space for all dimensions. If we specialize to the phase space of massless particles only, then our result agrees with twistors that were previously defined for $d \leq 6$ in another approach [25]. Even for $d \leq 6$ our twistors for the phase spaces other than the massless particle are all new structures. For a rather different approach to twistors for massive particles, which uses double the number of twistors compared to our formulas and only in $d = 4$, see [3],[26]-[30].

Beyond twistors for particles, we have also defined twistors for a phase space that includes also D-brane degrees of freedom. Including the D-branes may lead to some interesting applications of twistors, in particular for M-theory. The twistor action principle in Eq.(2.14) applies generally to twistors including D-branes for a generally complex Z in every dimension. If (pseudo)reality conditions are imposed on Z as mentioned after the table in the previous section, then for $d \leq 6$ we obtain only the particle phase space out of the twistor. In $d \geq 7$ there are automatically D-branes even with the (pseudo) reality conditions. However, if extra constraints are applied on Z, as detailed in section (II B), then again the degrees of freedom in Z are thinned down to only the particle phase space without D-branes in every dimension.

Quantization of twistors for any spin in four dimensions was discussed in section (I B). Here we suggested a free field theory in twistor space that describes any spinning particle. This could also lead to some interesting applications of twistors in field theory with interactions in four dimensions.

Generalizations of our results in many directions are possible. Some of these are already briefly described in recent papers [12][13], such as twistors for spinning particles, supertwistors in various dimensions, including compactified dimensions, and supertwistors for supersymmetric $\mathrm{AdS}_5\times\mathrm{S}^5$, $\mathrm{AdS}_4\times\mathrm{S}^7$, $\mathrm{AdS}_7\times\mathrm{S}^4$. We plan to give details of those structures in future publications.

It must be emphasized that in all cases the underlying theory is anchored in 2T-physics, and therefore by gauge fixing the $\mathrm{Sp}(2,R)$ gauge symmetry these twistors describe not only massless systems, but much more, as discussed in this paper and [7] with examples. Hence the twistors play a role in some kind of unification of 1T-physics systems via dualities, or via

higher dimensions with 2T, but in a way that is distinctly different than the Kaluza-Klein scheme, since there are no Kaluza-Klein excitations, but instead there is a web of dualities.

In this context it is also interesting that some parameters such as mass, moduli of some metrics, and some coupling constants for interactions, emerge from the higher dimensions as moduli while holographically projecting from $d+2$ dimensions down to d dimensions. Furthermore concepts such as time and Hamiltonian in 1T-physics are derived concepts that emerge either from 2T-physics and its gauge choices, or from the details of the twistor transform to 1T-physics systems.

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[1] I. Bars, B. Orcal, and M. Picon, in preparation, I. Bars and B. Orcal, in preparation.

[2] R. Penrose, "Twistor Algebra," J. Math. Phys. **8** (1967) 345; "Twistor theory, its aims and achievements, in Quantum Gravity", C.J. Isham et. al. (Eds.), Clarendon, Oxford 1975, p. 268-407; "The Nonlinear Graviton", Gen. Rel. Grav. **7** (1976) 171; "The Twistor Program," Rept. Math. Phys. **12** (1977) 65.

[3] R. Penrose and M.A. MacCallum, "An approach to the quantization of fields and space-time", Phys. Rept. **C6** (1972) 241; R. Penrose and W. Rindler, Spinors and space-time II, Cambridge Univ. Press (1986).

[4] I. Bars, C. Deliduman and O. Andreev, " Gauged Duality, Conformal Symmetry and Spacetime with Two Times" , Phys. Rev. **D58** (1998) 066004 [arXiv:hep-th/9803188]. For reviews of subsequent work see: I. Bars, " Two-Time Physics" , in the Proc. of the 22nd Intl. Colloq. on Group Theoretical Methods in Physics, Eds. S. Corney at. al., World Scientific 1999, [arXiv:hep-th/9809034]; " Survey of two-time physics," Class. Quant. Grav. **18**, 3113 (2001) [arXiv:hep-th/0008164]; " 2T-physics 2001," AIP Conf. Proc. **589** (2001), pp.18-30; AIP Conf. Proc. **607** (2001), pp.17-29 [arXiv:hep-th/0106021].

[5] I. Bars, "Conformal symmetry and duality between free particle, H-atom and harmonic oscillator", Phys. Rev. **D58** (1998) 066006 [arXiv:hep-th/9804028]; "Hidden Symmetries, $\mathrm{AdS}_d \times \mathrm{S}^n$,

and the lifting of one-time physics to two-time physics", Phys. Rev. **D59** (1999) 045019 [arXiv:hep-th/9810025].

[6] I. Bars, " 2T physics formulation of superconformal dynamics relating to twistors and supertwistors," Phys. Lett. B **483**, 248 (2000) [arXiv:hep-th/0004090].

[7] I. Bars and M. Picon, "Single twistor description of massless, massive, AdS, and other interacting particles," arXiv:hep-th/0512091.

[8] E. Witten, "Perturbative gauge theory as a string theory in twistor space", Commun. Math. Phys. **252** (2004) 189 [arXiv:hep-th/0312171]; "Parity invariance for strings in twistor space", hep-th/0403199.

[9] N. Berkovits, "An Alternative string theory in twistor space for N=4 Super Yang-Mills", Phys. Rev. Lett. **93** (2004) 011601 [arXiv:hep-th/0402045].

[10] N. Berkovits and L. Motl, "Cubic twistorial string field theory", JHEP **0404** (2004) 56, [arXiv:hep-th/0403187].

[11] N. Berkovits and E. Witten, "Conformal supergravity in twistor-string theory", JHEP **0408** (2204) 009, [arXiv:hep-th/0406051].

[12] I. Bars, "Twistor superstring in 2T-physics," Phys. Rev. **D70** (2004) 104022, [arXiv:hep-th/0407239].

[13] I. Bars, "Twistors and 2T-physics," AIP Conf. Proc. **767** (2005) 3 , [arXiv:hep-th/0502065].

[14] F. Cachazo, P. Svrcek and E. Witten, " MHV vertices and tree amplitudes in gauge theory", JHEP **0409** (2004) 006 [arXiv:hep-th/0403047]; " Twistor space structure of one-loop amplitudes in gauge theory", JHEP **0410** (2004) 074 [arXiv:hep-th/0406177]; "Gauge theory amplitudes in twistor space and holomorphic anomaly", JHEP **0410** (2004) 077 [arXiv:hep-th/0409245].

[15] For a review of Super Yang-Mills computations and a complete set of references see: F.Cachazo and P.Svrcek, "Lectures on twistor strings and perturbative Yang-Mills theory," PoS **RTN2005** (2005) 004, [arXiv:hep-th/0504194].

[16] I. Bars, "Two time physics with gravitational and gauge field backgrounds", Phys. Rev. **D62**, 085015 (2000) [arXiv:hep-th/0002140]; I. Bars and C. Deliduman, " High spin gauge fields and two time physics", Phys. Rev. **D64**, 045004 (2001) [arXiv:hep-th/0103042].

[17] I. Bars, "Lectures on Twistors", hep-th/0601091.

[18] I. Bars, " Hidden 12-dimensional structures in AdS_5 x S^5 and M^4 x R^6 supergravities," Phys. Rev. D **66**, 105024 (2002) [arXiv:hep-th/0208012]; " A mysterious zero in $AdS_(5)$ x S^5 supergravity," Phys. Rev. D **66**, 105023 (2002) [arXiv:hep-th/0205194].

[19] I. Bars, "Toy M-model", to be published. For a brief discussion see [21] and the second review paper in [4].

[20] I. Bars, " S Theory" , Phys. Rev. **D55** (1997) 2373 [arXiv:hep-th/9607112].

[21] I. Bars, C. Deliduman and D. Minic, " Lifting M-theory to Two-Time Physics" , Phys. Lett.

B457 (1999) 275 [arXiv:hep-th/9904063].

[22] P. West, “Hidden superconformal symmetry in M theory,” JHEP **0008** (2000) 007 [arXiv:hep-th/0005270].

[23] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo, J. Lukierski, Phys. Rev. Lett. **86** (2001) 4451 [arXiv:hep-th/0101113].

[24] I.A. Bandos, J.A. de Azcárraga, M. Picón and O. Varela, Phys. Rev. **D69** (2004), 085007 [arXiv:hep-th/0307106].

[25] I. Bengtsson and M. Cederwall, “Particles, Twistors And The Division Algebras,” Nucl. Phys. **B302** (1988) 81. M. Cederwall, “AdS twistors for higher spin theory,” AIP Conf. Proc. 767 (2005) 96 , [arXiv:hep-th/0412222].

[26] Z. Perj´es, Rep. Math. Phys. **12**, 193 (1977).

[27] L.P. Hughston, “Twistors and Particles”, Lecture Notes in Physics **97**, Springer Verlag, Berlin (1979).

[28] P.A. Tod, Rep. Math. Phys. **11**, 339 (1977).

[29] S. Fedoruk and V.G. Zima, “Bitwistor formulation of massive spinning particle,” arXiv:hep-th/0308154.

[30] A. Bette, J.A. de Azcarraga, J.Lukierski and C. Miquel-Espanya, “Massive relativistic particle model with spin and electric charge from two-twistor dynamics,” Phys. lett.B595, 491 (2004), [arXiv:hep-th/0405166]; J.A.de Azcarraga, A.Frydryszak, J.Lukierski and C.Miquel-Espanya, “Massive relativistic particle model with spin from free two-twistor dynamics and its quantization,” arXiv:hep-th/0510161.